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Métriques de Kähler-Einstein sur les compactifications de groupes

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Résumé

Le résultat principal de cette thèse est l'obtention d'une condition nécessaire et suffisante pour l'existence d'une métrique de Kähler-Einstein sur une compactification bi-équivariante lisse et Fano d'un groupe complexe réductif connexe. Ces variétés comprennent les variétés toriques et les compactifications magnifiques de groupes semisimples adjoints.

Dans la première partie de ce travail sont développés les outils nécessaires à l'étude de l'existence de métriques de Kähler-Einstein sur ces variétés. Nous calculons en particulier la Hessienne complexe d'une fonction $K \times K$-invariante sur la complexification d'un groupe compact K. Nous associons également, à toute métrique invariante à courbure positive sur un fibré linéarisé ample sur une compactification de groupe, une fonction convexe dont le comportement asymptotique est prescrit. Ceci est utilisé une première fois pour obtenir une formule pour l'invariant alpha d'un fibré en droite ample sur une compactification de groupe Fano. Cette formule est obtenue par le calcul des seuils log canoniques des métriques hermitiennes invariantes à courbure positive, et induit, dans le cas particulier des variétés toriques, un résultat obtenu auparavant, figurant dans l'article par ailleurs inclus en appendice de la thèse.

Nous prouvons ensuite le résultat principal en obtenant des estimées C^0 le long de la méthode de continuité, en se ramenant à une équation de Monge-Ampère réelle sur un cône. La condition obtenue est que le barycentre du polytope associé à la compactification de groupe, par rapport à la mesure de Duistermaat-Heckman, doit être dans une zone particulière du polytope. Cette condition peut être vérifiée sur les exemples, donne de nouveaux exemples de variétés de Kähler-Einstein Fano, et donne aussi un exemple qui n'admet aucun soliton de Kähler-Ricci. Nous calculons de plus la plus grande borne inférieure de Ricci lorsqu'il n'y a pas de métrique de Kähler-Einstein.

Abstract

The main result of this work is a necessary and sufficient condition for the existence of a Kähler-Einstein metric on a smooth and Fano bi-equivariant compactification of a complex connected reductive group. Examples of such varieties include wonderful compactifications of adjoint semisimple groups.

The tools needed to study the existence of Kähler-Einstein metrics on these varieties are developed in the first part of the work, including a computation of the complex Hessian of a $K \times K$-invariant function on the complexification of a compact group K. Another step is to associate to any non-negatively curved invariant hermitian metric on an ample linearized line bundle on a group compactification a convex function with prescribed asymptotic behavior. This is used a first time to derive a formula for the alpha invariant of an ample line bundle on a Fano group compactification. This formula is obtained through the computation of the log canonical thresholds of any non-negatively curved invariant hermitian metric, and gives the same result, for toric manifolds, as the one we obtained before, in an article that is included in this thesis as an appendix.

Then we prove the main result by obtaining C^0 estimates along the continuity method, using the tools developed to reduce to a real Monge-Ampère equation on a cone. The condition obtained is that the barycenter of the polytope associated to the group compactification, with respect to the Duistermaat-Heckman measure, lies in a certain zone in the polytope. This condition can be checked on examples, gives new examples of Fano Kähler-Einstein manifolds, and also gives an example that admits no Kähler-Ricci solitons. We also compute the greatest Ricci lower bound when there are no Kähler-Einstein metrics.

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Chapitre 1

Introduction et Résumé (Français)

1.1 Métriques de Kähler-Einstein Fano

1.1.1 Métriques de Kähler-Einstein

Une métrique Riemannienne g est dite *d'Einstein* lorsque sa courbure de Ricci $\mathrm{Ric}(g)$ vérifie $g = \lambda \mathrm{Ric}(g)$ pour une certaine constante réelle λ. Une métrique de *Kähler-Einstein* sur une variété complexe X est une métrique Riemannienne qui est à la fois de Kähler et d'Einstein. La donnée d'une métrique de Kähler g est équivalente à la donnée de sa forme de Kähler associée ω. De manière similaire, la courbure de Ricci d'une métrique de Kähler peut être considérée comme une $(1,1)$-forme que nous noterons $\mathrm{Ric}(\omega)$.

Décrivons cette forme localement. Une forme de Kähler ω sur X peut s'écrire sur une carte suffisamment petite comme $i\partial\overline{\partial}\varphi$ pour une certaine fonction lisse et strictement plurisousharmonique φ sur un ouvert de $\mathbb{C}^n$. Notons $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ la Hessienne complexe de φ, c'est-à-dire la matrice dont les coefficients sont les $\frac{\partial^2\varphi}{\partial z_i \partial \overline{z_j}}$. Rappelons qu'une fonction φ est *strictement plurisousharmonique* (psh) si sa Hessienne complexe est définie positive partout. La forme de Ricci de ω est alors définie dans cette carte par

$$\mathrm{Ric}(\omega) = i\partial\overline{\partial}\left(-\ln\det\mathrm{Hess}_{\mathbb{C}}(\varphi)\right).$$

Nous supposons dans la suite que la variété X est compacte. Il est important de noter que pour toute métrique de Kähler ω sur une variété de Kähler compacte X, la forme de Ricci de ω est dans une classe de cohomologie fixe qui dépend uniquement de X : la *première classe de Chern* $c_1(X)$ de X.

Considérons maintenant l'équation de Kähler-Einstein :

$$\mathrm{Ric}(\omega) = \lambda\omega$$

pour une certaine constante λ. Supposons que ω soit une forme de Kähler, solution de cette équation. Si λ est strictement négative, cela signifie qu'il y a une forme de Kähler $-\lambda\omega$ dans la classe $-c_1(X)$ et donc $c_1(X) < 0$. Si λ est égale à 0, alors $0 = \mathrm{Ric}(\omega) \in c_1(X)$ donc la première classe de Chern de X s'annule. De telle variétés sont dites de Calabi-Yau. Enfin si λ est strictement positive, cela signifie qu'il y a une forme de Kähler dans $c_1(X)$, donc que cette classe est positive, et la variété X est alors dite de *Fano*.

Cette discussion montre d'abord que l'équation de Kähler-Einstein ne peut pas avoir de solution lorsque la première classe de Chern de X n'est pas nulle ou de signe défini. De plus, cela montre qu'étant donnée une variété X susceptible d'admettre une métrique de Kähler-Einstein avec λ non nulle, l'étude se réduit au cas des métriques de Kähler dans la classe $c_1(X)$, ou $-c_1(X)$ selon le signe de λ. La constante λ, lorsqu'elle n'est pas nulle, peut être fixée à 1 ou -1, quitte à renormaliser la métrique.

Supposons maintenant que $\mathrm{Ric}(\omega)$ et $\lambda\omega$ soient dans la même classe. Nous allons expliquer comment l'équation de Kähler-Einstein se ramène à une EDP sur une fonction grâce au lemme du $\partial\overline{\partial}$. Soit ω_{ref} une métrique de référence fixée dans la classe de ω. Par le lemme du $\partial\overline{\partial}$, nous pouvons écrire, d'une part,

$$\omega = \omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi$$

et d'autre part,

$$\mathrm{Ric}(\omega_{\mathrm{ref}}) = \lambda\omega_{\mathrm{ref}} + i\partial\overline{\partial}f$$

où φ et f sont deux fonctions lisses sur X. L'équation de Kähler-Einstein peut alors s'écrire :

$$\begin{aligned}
\mathrm{Ric}(\omega) &= \lambda\omega \\
\mathrm{Ric}(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi) &= \lambda(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi) \\
&= \mathrm{Ric}(\omega_{\mathrm{ref}}) - i\partial\overline{\partial}f + \lambda i\partial\overline{\partial}\varphi
\end{aligned}$$

en travaillant localement, avec $\omega_{\mathrm{ref}} = i\partial\overline{\partial}\varphi_{\mathrm{ref}}$, nous obtenons

$$-\partial\overline{\partial}\ln\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}} + \varphi) = -\partial\overline{\partial}(\ln\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}}) + f - \lambda\varphi).$$

Toujours localement, cela est équivalent à

$$-\partial\overline{\partial}\ln\frac{\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}} + \varphi)}{\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}})} = -\partial\overline{\partial}(f - \lambda\varphi).$$

Mais le membre de gauche peut s'écrire $-\partial\overline{\partial}\ln\frac{\omega^n}{\omega_{\mathrm{ref}}^n}$ où n est la dimension de X, ce qui fait que les deux expressions sont définies globalement. Nous avons donc

$$\omega^n = e^{f+c-\lambda\varphi}\omega_{\mathrm{ref}}^n$$

pour une constante c (une fonction $\partial\overline{\partial}$-exacte sur X qui est compacte). La constante c est déterminée par le volume de ω^n, et peut être prise en compte dans la fonction f qui n'est pour l'instant définie qu'à une constante additive près. Finalement, nous obtenons

$$(\omega_{\text{ref}} + i\partial\overline{\partial}\varphi)^n = e^{f-\lambda\varphi}\omega_{\text{ref}}^n$$

et cette équation au dérivées partielles en ϕ est en fait équivalente à l'équation de Kähler-Einstein.

Le cas où $c_1(X) < 0$ a été résolu par Aubin dans [Aub76] : il existe toujours une métrique de Kähler-Einstein, et le cas où $c_1(X) = 0$ a été résolu par Yau. Plus généralement, Yau a prouvé le théorème (de Calabi-Yau) suivant, conjecturé par Calabi.

Théorème. *[Yau78] Soit* $[\omega_0]$ *la classe d'une forme de Kähler sur* X *et* θ *une forme représentant* $c_1(X)$*, alors il existe une métrique de Kähler* $\omega \in [\omega_0]$ *telle que* $\mathrm{Ric}(\omega) = \theta$.

1.1.2 Métriques de Kähler-Einstein Fano

Dans le cas Fano, l'équation de Kähler-Einstein n'admet pas toujours de solutions.

Obstructions

Une première obstruction, obtenue par Matsushima, est que la composante connexe de l'identité dans le groupe d'automorphismes d'une variété de Fano qui admet une métrique de Kähler-Einstein doit être un groupe réductif. Matsushima prouve aussi que le groupe des isométries holomorphes d'une métrique de Kähler-Einstein est aussi grand que possible.

Théorème. *[Mat57] Soit* X *une variété de Fano admettant une métrique de Kähler-Einstein. Alors* $\mathrm{Aut}^0(X)$ *est un groupe réductif complexe, et le groupe des isométries holomorphes d'une métrique de Kähler-Einstein est un sous-groupe compact maximal de* $\mathrm{Aut}^0(X)$.

Futaki a introduit ensuite un invariant intégral [Fut83], appelé à présent l'*invariant de Futaki*, qui est un caractère d'algèbres de Lie entre l'algèbre de Lie des champs de vecteurs holomorphes $\eta(X)$ et $\mathbb{R}$. Futaki a montré que ce caractère s'annule lorsque X admet une métrique de Kähler-Einstein. Remarquons que $\eta(X)$ est l'algèbre de Lie du groupe $\mathrm{Aut}^0(X)$, donc que le théorème de Matsushima implique que cette algèbre de Lie est réductive lorsqu'il existe une métrique de Kähler-Einstein.

La méthode de continuité

Une partie du travail accompli pour résoudre l'équation de Kähler-Einstein dans le cas $c_1(X) \leq 0$ peut encore être utilisée dans le cas Fano. En effet, l'outil principal pour résoudre ces cas est la méthode de continuité, qui demande d'obtenir des estimées a priori des solutions. La méthode de continuité consiste à considérer, à la place de l'équation de Kähler-Einstein seule, une famille d'équations indexées par $t \in [0,1]$, telle que l'équation pour $t=1$ soit l'équation de Kähler-Einstein. Le but est ensuite de prouver que l'ensemble des t tels qu'il existe une solution à l'équation correspondante est non-vide, ouvert et fermé. Par connexité cela implique qu'il existe une solution pour $t=1$.

Les résultats obtenus par Aubin et Yau peuvent être utilisés pour montrer que, dans la méthode de continuité que nous allons décrire, cet ensemble est ouvert. Des obstructions apparaissent cependant pour la fermeture de cet ensemble, mais il est encore possible d'utiliser leur travail pour se ramener à la recherche seulement d'estimées C^0 sur les solutions.

La famille d'équations apparaissant dans la méthode de continuité classique pour le cas Fano est la suivante :

$$(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi_t)^n = e^{f-t\varphi_t}\omega_{\mathrm{ref}}^n$$

où f est définie comme plus tôt, et où les métriques sont normalisées de manière à avoir $\lambda = 1$. Cela correspond, au niveau des 2-formes, et en notant $\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi_t$ par ω_t, à l'équation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_{\mathrm{ref}}.$$

À $t=0$, cette équation est résoluble par le théorème de Calabi-Yau. Pour montrer l'existence de métriques de Kähler-Einstein, il est suffisant d'obtenir des estimées $|\phi_t| \leq C$ pour les solutions, avec une constante C indépendante de t.

L'invariant α de Tian

Tian a déterminé dans [Tia87] une condition suffisante pour l'obtention d'estimées C^0. Celle-ci implique un invariant de la variété, appelé l'*invariant* α, qui encode les singularités possibles des métriques hermitiennes singulières à courbure positive sur $-K_X$. Si cet invariant satisfait l'inégalité $\alpha(X) > n/(n+1)$ alors il existe une métrique de Kähler-Einstein sur X. Dans le cas où un groupe compact agit sur la variété, cet invariant peut être raffiné en considérant uniquement les métriques hermitiennes invariantes par l'action de ce groupe. La condition suffisante d'existence reste la même pour cet invariant relatif à un groupe. Plus généralement, il est possible de définir cet invariant pour n'importe quel fibré en droite ample.

K-stabilité

L'avancée majeure récente dans le problème des métriques Kähler-Einstein Fano est la résolution de la conjecture de Yau-Tian-Donaldson par Chen, Donaldson et Sun [CDS15a, CDS15b, CDS15c] (voir aussi [Tia]). Ce résultat relie l'existence d'une métrique de Kähler-Einstein sur une variété de Fano à une condition algébro-géométrique de stabilité. Ce résultat fournit une condition nécessaire et suffisante d'existence. Pour obtenir des exemples de métriques de Kähler-Einstein cependant, cette condition n'est pas facile à vérifier en pratique, et n'est pas encore très bien comprise.

1.1.3 Exemples de variétés Kähler-Einstein Fano

En dimension un, le seul exemple de variété de Kähler compacte de Fano est la droite projective $\mathbb{P}^1$, qui admet pour métrique de Kähler-Einstein la métrique de Fubini-Study. Pour les surfaces de Fano, appelées surfaces de Del Pezzo, Tian [Tia90] a montré que l'existence d'une métrique de Kähler-Einstein dans ce cas est équivalente à l'annulation de l'invariant de Futaki, ou encore au fait que l'algèbre de Lie des champs de vecteurs holomorphes soit réductive. Ce n'est plus vrai en dimension plus grande, comme Tian l'a montré sur un exemple dans [Tia97]. Nous décrivons dans la suite quelques familles d'exemples de variétés de Kähler-Einstein de Fano.

Le premier exemple de variété de Kähler-Einstein Fano en n'importe quelle dimension est bien sûr l'espace projectif $\mathbb{P}^n$ muni de la métrique de Fubini-Study. Plus généralement, tout variété de Fano compacte homogène admet une métrique de Kähler-Einstein.

La condition suffisante donnée par Tian en termes d'invariant α est un outil très puissant qui a permis de trouver de nombreuses familles de métriques de Kähler-Einstein Fano (voir par exemple [BAC02, EP, Süß13]). Tian a introduit initialement cet invariant pour prouver qu'une hypersurface de dimension n et degré supérieur à $n-1$ admet une métrique de Kähler-Einstein.

Dans le cas des variétés toriques, c'est-à-dire des variétés de dimension n sur lesquelles le tore $(\mathbb{C}^*)^n$ agit avec une orbite ouverte dense, le premier résultat majeur d'existence a été obtenu par Batyrev et Selivanova [BS99] grâce à l'invariant α. Ils ont montré que les variétés toriques Fano symétriques, c'est-à-dire celle pour lesquelles il existe un groupe compact d'automorphismes ne laissant aucun champs de vecteurs non nul invariant, admettent une métrique de Kähler-Einstein.

Ce résultat ne suffit pas à résoudre le problème Kähler-Einstein pour les variétés toriques. Pour les petites dimensions ($n \leq 6$), toute variété torique Fano est soit symétrique, soit sans métrique de Kähler-Einstein. Mais à partir de la dimension 7, il y a des exemples de variétés torique Kähler-Einstein non-symétriques. Ces exemples ont été trouvés par Nill et Paffenholz [NP11], en utilisant le théorème de Wang et Zhu qui ont complétement résolu la question de l'existence de métriques de Kähler-Einstein sur les variétés toriques.

Théorème. *[WZ04] Soit X une variété torique lisse Fano. Alors X admet une métrique de Kähler-Einstein si et seulement si l'invariant de Futaki de X s'annule.*

L'invariant de Futaki d'une variété torique Fano est donné par le barycentre du polytope associé à cette variété, comme l'a montré Mabuchi [Mab87]. En particulier, l'invariant de Futaki s'annule si et seulement si le barycentre du polytope est l'origine.

Le barycentre d'un polytope avec des sommets entiers peut être calculé explicitement, donc ce critère peut être vérifié en pratique étant donné le polytope d'une variété torique. Nill and Paffenholz ont utilisé une classification des polytopes correspondant jusqu'à la dimension 8 pour obtenir un exemple dont le barycentre associé est l'origine mais tel que la variété associée ne soit pas symétrique. D'après le calcul par Song [Son05] des invariants α des variétés toriques Fano, non seulement les exemples ne sont pas symétriques, mais aussi le critère de Tian en termes d'invariant α n'aurait pas pu être utilisé sur ces variétés.

La méthode de démonstration de Wang et Zhu est d'obtenir directement des estimées C^0 le long de la méthode de continuité. Puisque nous suivrons la même stratégie qu'eux, nous expliquons le point de départ de leur méthode. Un ingrédient principal est d'utiliser le théorème de Matsushima pour montrer qu'une métrique de Kähler-Einstein sur une variété torique doit être invariante sous l'action du tore compact $(\mathbb{S}^1)^n \subset (\mathbb{C}^*)^n$ (à conjugaison près). Il est de plus possible de supposer que si la métrique de référence est invariante par le tore compact, alors toutes les solutions le long de la méthode de continuité seront aussi invariant par le tore compact.

En se restreignant ensuite à l'orbite ouverte $(\mathbb{C}^*)^n$, ils traduisent l'équation de Kähler-Einstein en une équation de Monge-Ampère réelle, impliquant une fonction réelle convexe dont le comportement asymptotique à l'infini est prescrit. Pour voir cela, la première remarque à faire est que le quotient de $(\mathbb{C}^*)^n$ par $(\mathbb{S}^1)^n$ est isomorphe à $\mathbb{R}^n$. Deuxièmement, l'équation de Kähler-Einstein peut être exprimée localement comme une équation de Monge-Ampère complexe en le potentiel local de la métrique, qui est une fonction psh. Troisièmement, dans un choix de coordonnées appropriées (logarithmiques), le Monge-Ampère complexe d'une fonction $(\mathbb{S}^1)^n$-invariante est égal au Monge-Ampère réel dans le quotient $\mathbb{R}^n$, et une fonction psh devient une fonction convexe sur le quotient. Finalement, la fonction convexe obtenue sur $\mathbb{R}^n$ doit satisfaire certaines conditions sur son comportement asymptotique si elle provient d'une métrique définie sur la variété toute entière. Dans ce cadre, Wang et Zhu parviennent à obtenir des estimées pour les solutions de telles équations de Monge-Ampère sur $\mathbb{R}^n$.

Une variété *presque-homogène* est une variété munie de l'action d'un groupe de Lie complexe avec une orbite dense. Par exemple, les variétés toriques sont presque-homogènes sous l'action du tore. Les premiers exemples de variétés de Kähler-Einstein Fano non homogènes furent obtenus par Sakane et Koiso [KS86, KS88] comme fibrés en $\mathbb{P}^1$ au-dessus d'autres variétés Kähler-Einstein Fano. Les exemples qu'ils exhibèrent ainsi étaient presque-homogènes.

Une généralisation des exemples donnés par Koiso et Skane a été étudiée par Podesta et Spiro : ils ont traité la question de l'existence de métriques de Kähler-Einstein sur des fibrés toriques homogènes [PS10]. Il s'agit de fibrés homogènes sur des variétés de drapeaux de groupes semisimples complexes, dont la fibre est une variété torique. Ils ont déterminé quand ces variétés sont Fano, et quand, dans ce cas, l'invariant de Futaki s'annule. Remarquons que ces variétés sont presque-homogènes pour un groupe réductif, et que le quotient de l'orbite ouverte par un sous-groupe compact maximal est encore isomorphe à $\mathbb{R}^r$, où cette fois r est la dimension de la fibre torique. Comme dans le cas torique, il est possible d'utiliser l'invariance sous l'action du groupe compact pour simplifier l'équation de Kähler-Einstein, et cela donne encore une équation de Monge-Ampère réelle sur $\mathbb{R}^r$, avec le même type de comportement asymptotique prescrit que dans le cas torique. En appliquant le travail de Wang et Zhu avec quelque modifications mineures, Podesta et Spiro concluent que leur variétés admettent une métrique de Kähler-Einstein si et seulement si leur invariant de Futaki s'annule.

Pour conclure cette liste non-exhaustive d'exemples, mentionnons le travail de Guan sur l'existence de métriques de Kähler-Einstein (et plus généralement de métriques canoniques) sur les variétés presque-homogènes de cohomogénéité un, ce qui signifie qu'un sous-groupe compact maximal agit avec une orbite de codimension un. Sur ces variétés, qu'il est difficile de classifier complétement, l'équation de Kähler-Einstein doit se traduire en un certain sens en une équation différentielle ordinaire grâce à l'action du groupe compact. Les derniers articles de Guan sur le sujet sont [Gua11a, Gua11b, Gua11c, Gua13].

1.1.4 Compactifications de groupes

Les variétés de Fano que nous étudierons dans cette thèse sont des compactifications de groupes. Soit G un groupe algébrique complexe réductif connexe (par exemple $GL_n(\mathbb{C})$, $SL_n(\mathbb{C})$, $(\mathbb{C}^*)^n$, etc.).

Une *variété torique compacte* Z de dimension r est une compactification du tore $T = (\mathbb{C}^*)^r$. Le fait que le tore T agisse sur la variété Z entière et que $T \subset Z$ soit une orbite pour la multiplication à gauche signifie que Z est en fait une compactification T-équivariante de T. En d'autre termes, il s'agit de la donnée d'un plongement j de T dans une variété compact Z telle que Z admette une action de T et que j soit une application équivariante, c'est-à-dire $j(t\cdot z) = t\cdot j(z)$. De manière équivalente, il est aussi possible d'utiliser l'action de T sur lui-même par multiplication à droite par l'inverse. Comme T est abélien, les deux actions sont équivalentes.

Si l'on considère un groupe réductif non abélien G, ces deux actions ne sont plus équivalentes. Nous pourrions toujours considérer des compactifications équivariante par la multiplication à gauche de G, ou par la multiplication à droite, mais nous allons ici supposer que les compactifications sont équivariantes simultanément pour les deux actions. Nous considérons donc des compactifications $G \times G$-équivariantes d'un groupe réductif complexe connexe G. Pour simplifier, nous appellerons dans ce texte ces variétés des *compactifications de G*, ou des *compactifications de groupes* lorsque le groupe n'est pas fixé.

Une des raisons principales pour considérer ces deux actions est que de telles variétés sont sphériques. Une variété X est dite *sphérique* lorsqu'elle est munie d'une action d'un groupe réductif G telle qu'un sous-groupe de Borel de G agisse sur X avec une orbite ouverte. Cela implique que G admet une orbite dense et ouverte dans X, cette orbite étant un espace homogène pour G. En particulier, une variété sphérique est presque-homogène. Ce sont les variétés presque homogènes pour lesquelles l'étude est la plus avancée, et pour lesquelles beaucoup de problèmes résolus pour les variétés toriques ont une chance d'avoir une solution similaire.

En fait, la plupart des exemples de variétés Kähler-Einstein Fano mentionnées précédemment sont des variétés sphériques. Il est facile de voir que les variétés compactes Fano homogènes sont sphériques, et nous avons vu que les variétés toriques l'étaient également. Les exemples étudiés par Podesta et Spiro font partie de la famille des variétés horosphériques (parmi les variétés sphériques), et contiennent à la fois les variétés homogènes et les variétés toriques.

Comme nous l'avons vu, les variétés toriques sont aussi des compactifications de groupes, pour les groupes réductifs abéliens. À l'opposé, les groupes réductifs les "moins abéliens" sont les groupes semisimples. Les exemples les plus connus de compactifications de groupes semisimples sont les compactifications magnifiques de groupes semisimples adjoints, construites par De Concini et Procesi [DCP83]. En résumé, ce sont des variétés lisses et Fano, construites comme l'adhérence de l'image de G dans une représentation irréductible $G \to GL_N(\mathbb{C})$ associée à un poids régulier et dominant.

Donaldson a suggéré dans son survey [Don08] d'étudier l'existence de métriques de Kähler-Einstein, et plus généralement, l'existence de métriques extrémales ou à courbure scalaire constante, pour les variétés sphériques. En effet, elles entrent dans la première catégorie de variétés présentée par Donaldson dans [Don08, Section 4] car elles sont sans multiplicité (multiplicity free). Remarquons que la classe plus petite des compactifications de groupes que nous considérons ici entre également dans la deuxième catégorie de variétés présentées dans [Don08, Section 4].

1.2 Résumé de la thèse

1.2.1 Chapitre 3

Le but du chapitre 3 est de donner les outils nécessaires pour travailler sur un groupe réductif complexe G, et donc sur l'orbite ouverte et dense d'une compactification de G. La première section fournit la définition d'un groupe réductif, et rappelle plusieurs outils usuels pour les étudier. Le système de racines Φ associé à un groupe réductif sera très important pour nous puisque nos résultats seront toujours exprimés en termes de ce système de racines, pour ce qui concerne l'orbite dense. La deuxième section se concentre sur l'action d'un sous-groupe compact maximal K sur G par multiplication à droite et à gauche.

Soit G un groupe réductif complexe connexe, et K un sous-groupe compact maximal de G. Choisissons T un tore maximal de G, tel que $T \cap K$ soit un tore maximal de K. Notons $\mathfrak{g}$ l'algèbre de Lie de G, $\mathfrak{k}$ celle de K et $\mathfrak{t}$ celle de T. Le fait que G soit réductif complexe est équivalent au fait que G soit isomorphe à la *complexification* de K. Au niveau des algèbres de Lie, on peut écrire $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$. Notons $\mathfrak{a}$ la sous-algèbre de Lie $i\mathrm{Lie}(T \cap K)$.

L'outil de base lorsque l'on considère l'action de K à gauche et à droite est la *décomposition* KAK : tout élément g de G s'écrit sous la forme $k_1 \exp(a) k_2$ où $k_1, k_2 \in K$ et $a \in \mathfrak{a}$. On peut même être plus précis. Soit Φ le système de racines de (G, T), et choisissons un système de racines positives Φ^+. Alors cela détermine une chambre de Weyl positive fermée $\overline{\mathfrak{a}^+}$ dans $\mathfrak{a}$. Dans la décomposition KAK, il est en fait possible d'imposer que $a \in \overline{\mathfrak{a}^+}$, et cet élément a est uniquement déterminé par g. Autrement dit, cette décomposition fournit un domaine fondamental pour l'action de $K \times K$ sur G : l'image par exp de $\overline{\mathfrak{a}^+}$.

Rappelons que $\mathfrak{a}$ est un espace vectoriel réel de dimension r le rang de G, c'est-à-dire la dimension (complexe) de T, sur lequel agit le groupe de Weyl $W = N_G(T)/T$ de (G, T). Le cône $\overline{\mathfrak{a}^+}$ est aussi un domaine fondamental pour l'action du groupe de Weyl sur $\mathfrak{a}$.

Notre objectif est d'étudier les fonctions $K \times K$ invariantes sur G. Soit ψ une telle fonction. Étant donnée la décomposition KAK, il est évident que la donnée de ψ est équivalente à la donnée de la fonction f définie sur $\mathfrak{a}$ par $f(a) = \psi(\exp(a))$. Azad et Loeb [AL92] ont montré que la fonction ψ est plurisousharmonique si et seulement si la fonction f est *convexe*. Ainsi l'étude des fonctions plurisousharmoniques $K \times K$-invariantes sur G se ramène à l'étude des fonctions convexes W-invariantes sur $\mathfrak{a}$.

Un autre outil, qui sera utilisé dans les chapitres ultérieurs, est la *formule d'intégration KAK*, adaptée à la décomposition KAK. Précisément, si dg est une mesure de Haar sur G, dx est une mesure de Lebesgue correctement normalisée sur $\mathfrak{a}$ (par le réseau des sous-groupes à un paramètre), et J est la fonction définie sur $\mathfrak{a}$ par $J(x) = \prod_{\alpha \in \Phi^+} \sinh^2(\alpha(x))$, alors il existe une constante C dépendant uniquement du choix de la mesure de Haar dg, telle que pour toute fonction ψ $K \times K$-invariante et dg-intégrable,

$$\int_G \psi dg = C \int_{\mathfrak{a}^+} f(x) J(x) dx.$$

Notre contribution dans ce chapitre est le calcul de la *Hessienne complexe* d'une fonction $K \times K$-invariante ψ sur G dans un choix de coordonnées adaptées en termes de la Hessienne réelle $\mathrm{Hess}_{\mathbb{R}}(f)$ de la fonction correspondante f sur $\mathfrak{a}$.

Théorème 1.1. *Soit ψ une fonction $K \times K$-invariante sur G, et f la fonction associée sur $\mathfrak{a}$. Alors dans un choix de coordonnées approprié et pour $a \in \mathfrak{a}^+$, la Hessienne complexe de ψ est diagonale par blocs, égale à :*

$$\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(f)(a) & 0 & & 0 \\ 0 & M_{\alpha_1}(a) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_p}(a) \end{pmatrix}$$

où les $(\alpha_i)_{i \in \{1,\dots,p\}}$ parcourent les racines positives de Φ et M_α est défini par :

$$M_\alpha(a) = \frac{1}{2}\alpha(\nabla f(a)) \begin{pmatrix} \coth(\alpha(a)) & i \\ -i & \coth(\alpha(a)) \end{pmatrix}$$

et ∇f est le gradient de f par rapport à un produit scalaire fixé qui étend la forme de Killing sur la partie semisimple de $\mathfrak{a}$.

Comme nous l'avons vu dans cette introduction, ce calcul sera très important pour exprimer l'équation de Kähler-Einstein sur l'orbite ouverte de la compactification. Plus précisément nous utiliserons le déterminant de la Hessienne, appelé le *Monge-Ampère complexe* et noté $\mathrm{MA}_{\mathbb{C}}(\psi)$.

Corollaire 1.1. *Soit ψ une fonction $K \times K$-invariante sur G, et f la fonction associée sur $\mathfrak{a}$. Alors dans un choix de coordonnées approprié, et pour $a \in \mathfrak{a}^+$, si r est le rang de G et p le nombre de racines positives, nous avons :*

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{4^{r+p}}\mathrm{MA}_{\mathbb{R}}(f)(a) \prod_{\alpha\in\Phi^+} \frac{\alpha(\nabla f(a))^2}{\sinh^2(\alpha(a))}$$

Précisons dans quelles coordonnées ce calcul est valable. Pour cela rappelons que l'algèbre de Lie $\mathfrak{g}$ admet une décomposition en espaces de racines

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha,$$

où chaque $\mathfrak{g}_\alpha = \{x \in \mathfrak{g};\ \mathrm{ad}(h)(x) = \alpha(h)x\ \forall h \in \mathfrak{t}\}$ est un sous-espace vectoriel complexe de dimension 1. En choisissant une base de $\mathfrak{t}$, et un générateur pour chaque $\mathfrak{g}_\alpha$, on obtient une base de $\mathfrak{g}$ adaptée à cette décomposition. Pour prendre en compte le groupe compact K, nous utilisons une variante de cette décomposition en espaces de racines sur $\mathfrak{k}$:

$$\mathfrak{k} = i\mathfrak{a} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha,$$

où $\mathfrak{k}_\alpha = \{x \in \mathfrak{k};\ \mathrm{ad}(h)^2(x) = \alpha(h)^2 x\ \forall h \in \mathfrak{t}\}$. Nous avons alors $\mathfrak{k}_\alpha \oplus i\mathfrak{k}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, et pouvons modifier la base précédente pour obtenir une base de $\mathfrak{g}$ qui est aussi une base réelle de $\mathfrak{k}$. Il est aussi possible d'imposer certaines relations entre les éléments de la base que nous ne précisons pas ici.

En notant $l_1, \ldots, l_n$ les éléments de la base choisie, nous obtenons des coordonnées locales autour de $\exp(a) \in G$ pour $a \in \mathfrak{a}$, données par :

$$(z_1, \ldots, z_n) \mapsto \exp(z_1 l_1 + \cdots + z_n l_n)\exp(a).$$

Nous calculons alors les coefficients de la Hessienne dans ces coordonnées locales, c'est-à-dire que pour toute paire d'éléments (l_1, l_2) de la base, nous calculons :

$$\left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_{z_1,z_2=0} \psi(\exp(z_1 l_1 + z_2 l_2)\exp(a)).$$

Le principe du calcul est d'obtenir une décomposition KAK suffisamment explicite sur l'argument $\exp(z_1 l_1 + z_2 l_2)\exp(a)$, pour utiliser la $K \times K$-invariance. Nous utilisons pour cela la formule de Baker-Campbell-Haussdorf, qui donne le logarithme d'un produit d'exponentielles sous forme d'une série en les arguments de ces exponentielles. Dans le calcul que nous faisons, nous n'avons besoin que des termes d'ordre deux, et le choix de la base adaptée permet d'effectuer le calcul du terme central dans la décomposition KAK, à l'ordre deux.

1.2.2 Chapitre 4

Le chapitre 4 est une introduction aux compactifications de groupe où nous nous concentrons sur le polytope associé à une polarisation d'une telle variété. Le polytope contient toute l'information sur la frontière de la compactification, et sur le fibré ample choisi. C'est une généralisation du polytope associé à une variété torique polarisée, et nous utilisons en fait cette correspondance usuelle pour présenter le cas des compactifications de groupes. En effet, étant donnée une compactification X de G, considérons T un tore maximal dans G et Z son adhérence dans X. Alors Z est une variété torique, contenant toute l'information de la compactification (si G est fixé).

De même, si L est un fibré ample $G \times G$-linéarisé sur X, la restriction $L|_Z$ de L à la sous-variété torique est encore un fibré ample, linéarisé sous l'action du normalisateur de T dans G. Le polytope P associé à la compactification polarisée (X, L) est alors défini comme le polytope associé à la variété torique polarisée $(Z, L|_Z)$.

Nous donnons quelques exemples de compactifications lisses et Fano de groupes. Il y a de nombreux tels exemples : tout groupe semisimple adjoint admet une compactification canonique avec une unique orbite fermée, sa compactification magnifique, et elle est lisse et Fano. Le polytope associé à une telle compactification polarisée par le fibré anticanonique est l'enveloppe convexe des images par le groupe de Weyl W de $2\rho + \sum_{i=1}^{r} \alpha_i$ où 2ρ est la somme des racines positives de Φ et les α_i sont les racines simples de Φ.

Nous considérons ensuite les métriques hermitiennes à courbure positive sur les fibrés en droites amples linéarisés sur les compactifications de groupes. À une telle métrique h nous associons une fonction plurisousharmonique ψ sur G, son potentiel par rapport à une trivialisation $G \times \{e\}$-équivariante s de L sur l'orbite isomorphe à G : $\psi(g) = -\ln(|s(g)|_h^2)$. Lorsque h est $K \times K$-invariante, la fonction ψ l'est également et cela détermine par, le chapitre 3, une fonction convexe sur $\mathfrak{a}^+$, appelée le *potentiel convexe* de h.

Nous prouvons que le fait que h soit une métrique à courbure positive sur une polarisation de X impose des conditions sur le comportement asymptotique de cette fonction convexe, et nous décrivons ces conditions en terme du polytope associé. Puisque la correspondance entre une métrique et son potentiel convexe est bijective, cela fournit une description de l'espace des métriques hermitiennes singulières $K \times K$-invariantes à courbure positive sur le fibré considéré.

Théorème 1.2. *Soit (X, L) une compactification lisse polarisée de G, de polytope associé P. Les métriques hermitiennes $K \times K$-invariante sur L, à courbure positive au sens des courants, sont en bijection avec les fonctions convexes W-invariante $\varphi : \mathfrak{a} \longrightarrow \mathbb{R}$ telles qu'il existe une constante $C_1 \in \mathbb{R}$ avec*

$$\varphi(x) \leq f_P(x) + C_1$$

sur $\mathfrak{a}$ où f_P est la fonction support du polytope $2P$. De plus, h est localement bornée si et seulement si il existe aussi une constante C_2 avec

$$f_P(x) + C_2 \leq \varphi(x) \leq f_P(x) + C_1.$$

La fonction φ associée à h est son potentiel convexe.

Dans le cas d'une métrique lisse à courbure strictement positive, le potentiel convexe φ est une fonction lisse et strictement convexe. Le changement de variable $p = \nabla\varphi(x)$ est donc bien défini et le théorème précédent assure que l'image par $\nabla\varphi$ de $\mathfrak{a}$ est l'intérieur du polytope $2P$ en identifiant $\mathfrak{a}$ et $\mathfrak{a}^*$. En prenant en compte l'action de W, on a même plus précisément que l'image par $\nabla\varphi$ de $\mathfrak{a}^+$ est l'intérieur du polytope $2P^+$.

En utilisant ce changement de variable conjointement avec la formule pour le Monge-Ampère complexe, on peut relier le degré du fibré ample L à une intégrale sur le polytope P : le volume de ce polytope pour la mesure de Duistermaat-Heckman.

Proposition 1.2. *Soit (X, L) une compactification lisse polarisée de G, correspondant au polytope P. Alors*

$$\deg(L) = C \int_{2P^+} \prod_{\alpha\in\Phi^+} (\alpha(p))^2 dp$$

pour une constante C qui dépend seulement du groupe G.

Ce résultat est cohérent avec la formule explicite obtenue par Kazarnovskii [Kaz87] et Brion [Bri89], mais nous ne déterminons pas dans notre calcul la constante C explicitement, puisque nous n'en avons pas besoin dans la suite.

1.2.3 Chapitre 5

Dans le chapitre 5, nous calculons l'invariant α d'un fibré ample L sur une compactification de groupe X Fano. Plus précisément, nous calculons l'invariant α par rapport à l'action d'un sous-groupe compact maximal $K \times K$ de $G \times G$. Ce résultat est obtenu en calculant le seuil log canonique de n'importe quelle métrique hermitienne $K \times K$-invariante h sur L à courbure positive au sens des courants, en termes d'un corps convexe associé à h.

Notons P le polytope associé à L, Q le polytope associé à $-K_X$, et H l'enveloppe convexe des images par le groupe de Weyl W de la somme des racines positives 2ρ.

Soit h une métrique hermitienne $K\times K$-invariante h sur L à courbure positive au sens des courants. Le corps convexe $N(h)$ associé à h, que l'on appelle le *corps de Newton* de h, est le domaine de la transformée de Legendre-Fenchel du potentiel convexe φ de h :

$$N(h) = \{m \in \mathfrak{a}^*; \exists C, \forall x \in \mathfrak{a},\ \varphi(x) - m(x) \geq C\}.$$

L'information dont nous avons besoin sur le comportement asymptotique de φ se traduit en fonction de ce corps convexe. En particulier, les conditions obtenues au chapitre précédent se traduisent par le fait que $N(h)$ est contenu dans $2P$ où P est le polytope associé à L, avec égalité pour les métriques localement bornées. L'avantage de travailler avec ces corps de Newton est qu'ils sont bien adaptés pour utiliser la décomposition en éventail donnée par la sous-variété torique Z.

La méthode pour calculer le seuil log canonique est de se ramener à un critère d'intégrabilité sur l'orbite ouverte G, puis sur $\mathbb{R}^r$ par la formule d'intégration KAK. À ce point, nous devons déterminer quand l'exponentielle d'une fonction concave est intégrable par rapport à la mesure de potentiel J contre la mesure de Lebesgue. En utilisant le critère d'intégrabilité obtenu par Guenancia dans une preuve analytique du calcul par Howald des seuils log canoniques d'idéaux monomiaux, nous obtenons un tel critère en fonction du corps de Newton.

Le critère permet d'obtenir l'expression du seuil log canonique.

Théorème 1.3. *Soit h une métrique hermitienne $K \times K$-invariante h sur L à courbure positive au sens des courants. Alors*

$$\mathrm{lct}(h) = \sup\{c > 0; 2H + 2cP \subset cN(h) + 2Q\},$$

où $N(h)$ est le corps de Newton de h.

Pour obtenir une expression de l'invariant α, nous déterminons quelles sont les métriques dont le seuil log canonique est le plus petit. Il s'agit des métriques dont le potentiel convexe est linéaire. Tout ceci donne le résultat suivant.

Théorème 1.4. *L'invariant α de L relatif à l'action de $K \times K$ est donné par la formule :*

$$\alpha_{K\times K}(L) = \sup\{c > 0; c(P + (-P^W)) \subset Q \ominus H\},$$

où P^W est l'ensemble des points W-invariants de P et $\ominus$ est la soustraction de Minkowski entre deux ensembles convexes.

En particulier, si le groupe G est semisimple, ou s'il y a suffisamment de symétries supplémentaires, il y a une métrique dont le seuil log canonique est égal à l'invariant alpha : celle dont le potentiel convexe est la fonction nulle.

Corollaire 1.3. *Supposons que G soit un groupe semisimple. Alors*

$$\alpha_{K\times K}(L) = \sup\{c \geq 0; cP \subset Q \ominus H\}.$$

Il s'agit aussi du rayon inscrit de $Q \ominus H$ par rapport à P.

Enfin, nous calculons l'invariant α pour quelques exemples de compactifications de groupes.

Corollaire 1.4. *Soit X la compactification magnifique de $\mathrm{PSL}_{n+1}(\mathbb{C})$, alors*

$$\alpha_{K\times K}(-K_X) = \frac{1}{1 + \lceil \frac{n}{2} \rceil (\lfloor \frac{n}{2} \rfloor + 1)}.$$

Cet exemple montre que de telles variétés ne satisfont pas la condition de Tian en général. En fait, il n'est pas difficile de se convaincre qu'il ne sera jamais satisfait pour les compactifications magnifiques de groupes semisimples adjoints sans facteur de rang un. Nous devons donc utiliser une autre méthode.

La formule que nous obtenons s'applique en particulier aux variétés toriques et permet de retrouver le calcul d'invariant α sur les variétés toriques, obtenu précédemment dans [Del15]. Le texte de cet article est également reproduit dans l'appendice A.

1.2.4 Chapitre 6

Le chapitre 6 contient le résultat principal de la thèse. Nous déterminons dans ce chapitre une condition nécessaire et suffisante d'existence d'une métrique de Kähler-Einstein sur une compactification de groupe. Cette condition est de plus calculable numériquement sur les exemples en termes du polytope associé à la compactification.

Soit G un groupe réductif connexe, X une compactification lisse et Fano de G, de polytope associé P. Soit Φ le système de racines de G, Φ^+ un choix de racines positives. Notons P^+ l'intersection de P avec la chambre de Weyl positive, 2ρ la somme des racines positives, et Ξ l'intérieur relatif du cône engendré par les racines simples. Nous identifions $\mathfrak{a}$ et son dual par le choix d'un produit scalaire, et notons dp la mesure de Lebesgue sur $\mathfrak{a}$ normalisée par le réseau des caractères.

Théorème 1.5. *Il existe une métrique de Kähler-Einstein sur X si et seulement si le barycentre*

$$\mathrm{bar}_{DH}(P^+) = \left(\int_{P^+} p \prod_{\alpha\in\Phi^+} (\alpha(p))^2 dp \right) \left(\int_{P^+} \prod_{\alpha\in\Phi^+} (\alpha(p))^2 dp \right)^{-1}$$

de P^+ par rapport à la mesure de Duistermaat-Heckman est dans $2\rho + \Xi$.

Lorsque G est semisimple, Ξ est le cône ouvert engendré par les racines simples de Φ. Lorsque G n'est pas semisimple, la dimension de Ξ est strictement inférieure au rang r de G. En particulier, lorsque G est un tore, ρ est l'origine et $\Xi = \{0\}$, de sorte que l'on retrouve le critère usuel pour les variétés toriques. En effet, dans ce cas la mesure de Duistermaat-Heckman est la mesure de Lebesgue sur $P = P^+$, et le critère signifie que le barycentre de P est l'origine.

Nous donnons ensuite quelques exemples de calculs et obtenons de nouveaux exemples de variétés de Kähler-Einstein Fano, mais aussi un exemple de compactification d'un groupe semisimple qui n'admet aucune métrique de Kähler-Einstein.

Exemple 1.5. Les compactifications magnifiques des groupes semisimples irréductibles de rang deux admettent des métriques de Kähler-Einstein.

Ces exemples sont de nouveaux exemples de métriques de Kähler-Einstein. Par exemple, la connaissance de la composante connexe de l'identité de leur groupe d'automorphismes (voir au chapitre 4) permet de voir qu'ils ne sont ni homogènes, ni toriques.

L'exemple suivant, qui n'admet pas de métrique de Kähler-Einstein, montre en particulier que l'annulation de l'invariant de Futaki sur une compactification de groupe ne suffit pas à assurer l'existence d'une métrique de Kähler-Einstein, alors que c'était le cas pour les classes de variétés sphériques déjà étudiées (homogènes, toriques ou horosphériques).

Exemple 1.6. L'éclaté de la compactification magnifique de $Sp(4)$ en l'orbite fermée est une compactification lisse de Fano qui n'admet pas de métrique de Kähler-Einstein.

Pour prouver le résultat, la première étape est de traduire l'équation de Kähler-Einstein, et plus généralement l'équation de la méthode de continuité, restreinte à l'orbite ouverte comme une équation de Monge-Ampère réelle sur un cône de $\mathbb{R}^r$. Le calcul du Monge-Ampère complexe effectué au chapitre 3 permet de le faire, et nous obtenons que le potentiel convexe u_t d'une solution $K \times K$-invariante de la méthode de continuité au temps t vérifie

$$\mathrm{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \alpha(\nabla u_t(x))^2 = e^{-(tu_t+(1-t)u_{\mathrm{ref}})(x)} J(x),$$

où u_{ref} est le potentiel convexe de la métrique de référence choisie, et J est la fonction apparaissant dans la formule d'intégration KAK et dans l'expression du Monge-Ampère complexe.

Pour prouver l'existence, il suffit d'obtenir une borne supérieure indépendante de t sur $u_t - u_{\mathrm{ref}}$. Nous obtiendrons cette borne en étudiant la fonction $\nu_t = tu_t + (1-t)u_{\mathrm{ref}} - \ln J$, définie sur $\mathfrak{a}^+$. Cette fonction est strictement convexe, et propre au sens où $\nu_t(x)$ tends vers $+\infty$ lorsque $|x_t|$ tends vers l'infini où que x_t s'approche d'un des murs de la chambre de Weyl $\mathfrak{a}^+$. Elle admet donc un minimum strict m_t en un unique point x_t de $\mathfrak{a}^+$. Notre but est d'obtenir des estimées sur m_t et $|x_t|$, qui, combinées aux informations sur le comportement asymptotique des potentiels convexes (issues du chapitre 4), permettront d'obtenir les estimées sur $u_t - u_{\mathrm{ref}}$.

Pour les estimées sur la valeur m_t du minimum, nous utilisons exactement le même procédé que Wang et Zhu dans [WZ04], et peu de choses sont à modifier. Ce procédé repose sur l'utilisation du principe de comparaison pour les équations de Monge-Ampère réelle, accompagné d'un résultat de John sur les corps convexes. Un petit argument supplémentaire permet d'obtenir une borne inférieure indépendante de t sur la rapidité à laquelle croît ν_t (au moins linéairement) à partir de ce minimum :

$$\nu_t(x) \geq \kappa|x - x_t| - C$$

pour κ et C deux constantes indépendantes de t.

Nous nous concentrons ensuite sur le point où le minimum est atteint x_t. Nous supposons qu'il n'existe pas de métrique Kähler-Einstein, et donc que $|x_t|$ n'est pas borné. Quitte à prendre une sous-suite, on peut supposer que $|x_t|$

tends vers $+\infty$, dans une direction donnée ξ. Pour en déduire que la condition de notre théorème n'est pas satisfaite, nous utilisons l'annulation :

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi} e^{-\nu_t} dx = 0,$$

dont la preuve, élémentaire, repose sur le comportement de ν_t aux bords de $\mathfrak{a}^+$. La partie la plus technique de ce travail est alors de traduire cette annulation, lorsque $|x_t|$ tends vers $+\infty$ en une information sur le polytope, en utilisant d'une part l'information sur la croissance de ν_t qui permet de voir que $e^{-\nu_t}dx$ est une mesure dont le poids est essentiellement concentrée près de x_t, et d'autre part le changement de variable $p = \nabla u_t$ pour se ramener au polytope $2P^+$. Finalement, nous traduisons l'annulation par

$$t_\infty(\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi) = (t_\infty - 1)(v - 4\rho)(\xi),$$

où t_∞ est la limite des t pour lesquels $|x_t|$ est fini, et v est la fonction support du polytope $2P$. Ceci permet d'obtenir la condition suffisante de notre résultat.

Pour la condition nécessaire,nous l'obtenons également grâce à l'annulation de l'intégrale précédente, appliquée à $\nu = u - \ln J$ où u est le potentiel convexe d'une métrique Kähler-Einstein.

Rappelons que dans le cas torique, la condition du théorème est équivalent à l'annulation de l'invariant de Futaki. Dans le cas où G n'est pas un tore, ce n'est plus le cas. En particulier, si X est la compactification magnifique d'un groupe semisimple, l'annulation de l'invariant de Futaki ne donne aucune information. C'est pour remplacer cela que nous avons utilisé le barycentre du polytope P^+ par rapport à la mesure de Duistermaat-Heckman, qui est un invariant des compactifications de groupes. Si G n'est pas un tore, la condition n'est pas l'annulation de cet invariant, mais le fait que le barycentre soit dans une certaine zone du polytope. De manière équivalente, comme l'annulation du Futaki est équivalente à l'annulation d'un nombre fini d'intégrales, ici, nous obtenons un ensemble fini d'inégalités et d'égalités à satisfaire par des intégrales.

Pendant la preuve de notre critère, nous calculons également la plus grande borne inférieure de Ricci de la variété lorsqu'elle n'admet pas de métrique de Kähler-Einstein. C'est le temps maximal d'existence d'une solution dans la méthode de continuité, mais aussi la borne supérieure des $t < 1$ tels qu'il existe une forme de Kähler ω dans $c_1(X)$ avec $\mathrm{Ric}(\omega) \geq t\omega$ [Szé11].

Théorème 1.6. *Supposons qu'il n'y a pas de métriques de Kähler-Einstein sur X et soit $R(X)$ la plus grande borne inférieure de Ricci de X. Alors*

$$R(X) = \sup\left\{ t < 1; \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(P^+)) + 2\rho \in (P^+ + (-\Xi)) \right\}.$$

Example 1.7. Lorsque X est la variété de l'exemple 2.10 ; nous avons calculé la valeur exacte de la plus grande borne inférieure de Ricci, qui est :

$$R(X) = \frac{1046175339}{1236719713} \simeq 0.8459\ldots$$

Chapter 2

Introduction

2.1 Fano Kähler-Einstein metrics

2.1.1 Kähler-Einstein metrics

A Riemannian metric g is Einstein if its Ricci curvature $\mathrm{Ric}(g)$ satisfies $g = \lambda \mathrm{Ric}(g)$ for some constant λ. A Kähler-Einstein metric on a complex manifold X is a Riemannian metric that is both Kähler and Einstein. The data of a Kähler metric g is equivalent to the data of its associated Kähler form ω. In a similar way, the Ricci curvature of a Kähler metric can be considered as a $(1,1)$-form that we denote by $\mathrm{Ric}(\omega)$.

Let us describe this form locally. A Kähler form ω on X can be written on a sufficiently small local chart as $i\partial\overline{\partial}\varphi$ for some smooth and strictly plurisubharmonic function φ on an open subset of $\mathbb{C}^n$. Let $\mathrm{Hess}_{\mathbb{C}}(\varphi)$ denote the complex Hessian of φ, *i.e.* the matrix whose coefficients are the $\frac{\partial^2\varphi}{\partial z_i \partial \overline{z_j}}$. Recall that a function φ is strictly plurisubharmonic (psh) if its complex Hessian is positive definite everywhere. The Ricci form of ω is then defined in this chart by

$$\mathrm{Ric}(\omega) = i\partial\overline{\partial}\left(-\ln\det \mathrm{Hess}_{\mathbb{C}}(\varphi)\right).$$

From now on we will assume that the manifold X is compact. The main observation to do is that for any Kähler metric ω on the compact Kähler manifold X, the Ricci form of ω lies in a fixed cohomology class depending only on X, which is the first Chern class $c_1(X)$ of X.

Now consider the Kähler-Einstein equation, in terms of forms:

$$\mathrm{Ric}(\omega) = \lambda\omega$$

for some constant λ. Assume that ω is a Kähler form, solution to this equation. If λ is negative, that means that there is a Kähler form $-\lambda\omega$ in the class $-c_1(X)$ which means $c_1(X) < 0$. If λ is zero, then that means that $0 = \mathrm{Ric}(\omega) \in c_1(X)$ and thus that the first Chern class vanishes. Manifolds satisfying this are called Calabi-Yau manifolds. Finally if λ is positive, that means that there is a Kähler form in $c_1(X)$, *i.e.* that this class is positive, and the manifold X is then called Fano.

This discussion shows first that the Kähler-Einstein equation cannot have a solution when the first Chern class of the complex manifold X is not definite or zero. Furthermore, it shows that given a manifold X where there could be a Kähler-Einstein metric with non zero constant λ, one has only to consider Kähler metrics in the class $c_1(X)$, or $-c_1(X)$ depending on the sign of λ. The constant λ, when it is not zero, can be normalized to either 1 or -1.

Assume now that $\mathrm{Ric}(\omega)$ and $\lambda\omega$ are in the same class. We will explain how the Kähler-Einstein equation reduces to a PDE on a function thanks to the $\partial\overline{\partial}$-lemma. Let ω_{ref} be a fixed reference Kähler metric in the class of ω. By the $\partial\overline{\partial}$-lemma we can write, on one hand

$$\omega = \omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi$$

and on the other hand

$$\mathrm{Ric}(\omega_{\mathrm{ref}}) = \lambda\omega_{\mathrm{ref}} + i\partial\overline{\partial}f$$

where φ and f are two smooth functions on X. The Kähler-Einstein equation can then be rewritten:

$$\begin{aligned}
\mathrm{Ric}(\omega) &= \lambda\omega \\
\mathrm{Ric}(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi) &= \lambda(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi) \\
&= \mathrm{Ric}(\omega_{\mathrm{ref}}) - i\partial\overline{\partial}f + \lambda i\partial\overline{\partial}\varphi
\end{aligned}$$

working locally with $\omega_{\mathrm{ref}} = i\partial\overline{\partial}\varphi_{\mathrm{ref}}$, we get

$$-\partial\overline{\partial}\ln\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}} + \varphi) = -\partial\overline{\partial}(\ln\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}}) + f - \lambda\varphi).$$

Still locally, this is equivalent to

$$-\partial\overline{\partial}\ln\frac{\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}} + \varphi)}{\det\mathrm{Hess}_{\mathbb{C}}(\varphi_{\mathrm{ref}})} = -\partial\overline{\partial}(f - \lambda\varphi).$$

But we can write the left hand side as $-\partial\overline{\partial}\ln\frac{\omega^n}{\omega_{\mathrm{ref}}^n}$ where n is the dimension of X, and then both expressions are valid globally. So we have

$$\omega^n = e^{f+c-\lambda\varphi}\omega_{\mathrm{ref}}^n$$

for some constant c (a $\partial\overline{\partial}$-exact function on X which is compact). The constant c is determined by the volume of ω^n, and can be absorbed in f which we defined only up to an additive constant. Finally we can write

$$(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi)^n = e^{f-\lambda\varphi}\omega_{\mathrm{ref}}^n$$

and this PDE in ϕ is in fact equivalent to the Kähler-Einstein equation.

The case when $c_1(X) < 0$ was solved by Aubin in [Aub76]: there always exists a Kähler-Einstein metric, and the case when $c_1(X) = 0$ was solved by Yau. In fact the following more general (Calabi-Yau) Theorem, conjectured by Calabi, was proved by Yau

Theorem. *[Yau78] Let $[\omega_0]$ be the class of a Kähler form on X and θ a form representing $c_1(X)$, then there exists a Kähler metric $\omega \in [\omega_0]$ such that* $\mathrm{Ric}(\omega) = \theta$.

2.1.2 Fano Kähler-Einstein metrics

In the Fano case, the Kähler-Einstein equation is not always solvable.

Obstructions

The first main obstruction, proved by Matsushima, shows that the connected component containing the identity of the automorphism group of a Fano manifold admitting a Kähler-Einstein metric must be a reductive group. It gives even more as it says that the group of holomorphic isometries of a Kähler-Einstein metric is as big as possible.

Theorem. *[Mat57] Let X be a Fano manifold and assume that it admits a Kähler-Einstein metric. Then the connected component of the identity in the automorphism group $\mathrm{Aut}^0(X)$ is a reductive complex algebraic group, and the group of holomorphic isometries of a Kähler-Einstein metric is a maximal compact subgroup of $\mathrm{Aut}^0(X)$.*

Futaki then found an integral invariant [Fut83], now called the *Futaki invariant*, which is a Lie character from the Lie algebra of holomorphic vector fields $\eta(X)$ of X to $\mathbb{R}$. Futaki proved that it vanishes whenever X admits a Kähler-Einstein metric. Remark that the Lie algebra of holomorphic vector fields is the Lie algebra of the Lie group $\mathrm{Aut}^0(X)$. The previous obstruction thus implies that $\eta(X)$ is a reductive Lie algebra.

The continuity method

On the positive side, parts of the work that has been done for the resolution of the Kähler-Einstein equation in the case $c_1(X) \leq 0$ can still be used in the Fano case. Namely these cases were solved using a continuity method and deriving a priori estimates on the solutions. The continuity method consists in considering instead of the Kähler-Einstein equation alone a family indexed by $t \in [0,1]$ of equations, where the equation at $t = 1$ is the Kähler-Einstein equation. The aim is to prove that the set of t such that there exists a solution is non-empty, open and closed. Connexity then implies that there is a solution at time $t = 1$.

The work of Aubin and Yau can be used to show that, in the continuity method that we will next describe, we have openness. The closedness is where there are obstructions, but we can use their work to reduce to determining a priori C^0 estimates on the solutions.

The family of equations in the (usual) continuity method for the Fano case is the following:

$$(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi_t)^n = e^{f - t\varphi_t}\omega_{\mathrm{ref}}^n$$

where f is defined as above, and we normalized the metrics to get $\lambda = 1$. It corresponds, at the level of two forms and denoting $\omega_{\mathrm{ref}} + i\partial\overline{\partial}\varphi_t$ by ω_t, to the equation

$$\mathrm{Ric}(\omega_t) = t\omega_t + (1-t)\omega_{\mathrm{ref}}.$$

The solvability at $t = 0$ is a consequence of the Calabi-Yau theorem. As we just said, it is enough to get a priori estimates $|\phi_t| \leq C$ for the solutions with a constant C independent of t to get existence of Kähler-Einstein metrics.

Tian's α-invariant

Tian obtained in [Tia87] a sufficient condition to get C^0 estimates. It involves an invariant of the manifold, $\alpha(X)$, called the α-invariant, that encodes the possible singularities of singular hermitian metrics with non negative curvature on $-K_X$. If this invariant satisfies $\alpha(X) > n/(n+1)$ then there exists a Kähler-Einstein metric on X. In the case of a manifold which admits an action of a compact group, this invariant can be refined by considering only metrics invariant under this group action. The sufficient condition for the existence of Kähler-Einstein metric remain the same. More generally, one can define the alpha invariant for any ample line bundle on a complex manifold X.

K-stability

The biggest advance in the Fano Kähler-Einstein problem in recent years was the resolution of the Yau-Tian-Donaldson conjecture by Chen-Donaldson-Sun [CDS15a, CDS15b, CDS15c] (see also [Tia]). This result relates the existence of a Kähler-Einstein metric on a Fano manifold with a condition of algebro-geometric stability. It gives a necessary and sufficient condition for the existence. Unfortunately for the purpose of finding examples of Kähler-Einstein metrics, this condition is not easy to check in practice for most manifolds, and is not yet very well understood.

2.1.3 Examples of Fano Kähler-Einstein manifolds

In dimension one, the only example of compact Kähler manifold that is Fano is the projective line $\mathbb{P}^1$, which admits as Kähler-Einstein metric the Fubini-Study metric. For Fano surfaces, also called Del Pezzo surfaces, the Kähler-Einstein problem was solved by Tian [Tia90] who showed that in this case the existence of a Kähler-Einstein metric is equivalent to the vanishing of the Futaki invariant, or even to the fact that the Lie algebra of holomorphic vector fields is reductive. This is no longer true in higher dimensions as Tian showed by an example in [Tia97]. Let us describe some of the largest families of examples of Kähler-Einstein Fano manifolds.

The first example of Fano Kähler-Einstein manifold in any dimension is of course the projective space $\mathbb{P}^n$ equipped with the Fubini-Study metric. More generally, any Fano compact homogeneous manifold admits a Kähler-Einstein metric.

The sufficient condition given by Tian in terms of α-invariant is a very powerful tool that allowed to find many families of examples of Fano Kähler-Einstein metrics (see *e.g.* [BAC02, EP, Süß13]). For example, Tian introduced this invariant to prove that a Fermat hypersurface of dimension n with degree greater than $n-1$ admits a Kähler-Einstein metric.

In the case of toric manifolds, *i.e.* manifolds of dimension n equipped with an action of the torus $(\mathbb{C}^*)^n$ with an open and dense orbit, the first major existence result was proved by Batyrev and Selivanova [BS99] using the α-invariant. They showed that all Fano toric manifolds that were *symmetric*, *i.e.* for which there exists a compact subgroup of automorphisms leaving no nonzero holomorphic vector field invariant, admitted a Kähler-Einstein metric.

This did not solve however the Kähler-Einstein problem for toric manifolds. Indeed, for small dimensions ($n \leq 6$), any Fano toric manifolds either admits no Kähler-Einstein metric or is symmetric. Starting from dimension 7, there are non-symmetric Kähler-Einstein toric manifolds. The examples were found by Nill and Paffenholz [NP11] with computer assistance, by using the theorem of Wang and Zhu who completely solved the Kähler-Einstein problem for toric manifolds.

Theorem. *[WZ04] Let X be a Fano toric manifold. Then X admits a Kähler-Einstein metric if and only if the Futaki invariant of X vanishes.*

As Mabuchi proved [Mab87], the Futaki invariant of a Fano toric manifold is given by the barycenter of the polytope associated to the manifold. In particular, the Futaki invariant vanishes if and only if the barycenter of the polytope is the origin.

The barycenter of a polytope with integral vertices can be computed, so the criterion can be checked in practice once the polytope of a toric manifold is given. Nill and Paffenholz used a classification of the corresponding polytopes up to dimension 8 to find an example of polytope of a Fano toric manifold with barycenter the origin but not symmetric. By Song's computation of the α-invariants of Fano toric manifolds [Son05], not only the examples were non-symmetric but also Tian's criterion in terms of α-invariant could not be used to get existence of a Kähler-Einstein metric.

Wang and Zhu's method of proof is to get directly C^0 estimates along the continuity method. Since our work follows the same strategy as they do, let us explain the starting point of their method. One main ingredient is to use Matsushima's theorem to derive that a Kähler-Einstein metric on a toric manifold must be invariant under the action of the compact torus $(\mathbb{S}^1)^n \subset (\mathbb{C}^*)^n$. One can further assume that, if we start from a compact torus invariant reference metric in the continuity method, all the solutions for any t are invariant under the compact torus.

Then restricting to the open $(\mathbb{C}^*)^n$ orbit, they translate the Kähler-Einstein equation as a real Monge-Ampère equation on $\mathbb{R}^n$ involving a convex real function with prescribed behavior at infinity. To see this, the first remark is that the quotient of $(\mathbb{C}^*)^n$ by $(\mathbb{S}^1)^n$ is isomorphic to $\mathbb{R}^n$. Secondly, the Kähler-Einstein equation can be locally expressed as a complex Monge-Ampère equation in the local potential of the metric, which is a psh function. Thirdly, in an appropriate choice of coordinates (logarithmic coordinates), the complex Monge-Ampère of a $(\mathbb{S}^1)^n$-invariant function is equal to the real Monge-Ampère in the $\mathbb{R}^n$ quotient coordinates, and a psh function becomes a convex function on the quotient. Finally the convex function obtained on $\mathbb{R}^n$ must satisfy some asymptotic behavior conditions if it comes from a metric that extends from the open orbit to the whole manifold. In this setting, Wang and Zhu are then able to obtain estimates for the solutions of such real Monge-Ampère equations on $\mathbb{R}^n$.

An *almost-homogeneous manifold* is a manifold equipped with an action of a complex Lie group with a dense orbit. For example, toric manifolds are almost-homogeneous under the torus action. The first examples of non homogeneous Fano Kähler-Einstein manifolds were obtained by Sakane and Koiso [KS86, KS88] as $\mathbb{P}^1$-bundles over other Fano Kähler-Einstein manifolds. The examples they exhibited were almost-homogeneous.

As a generalization of the examples given by Koiso and Sakane, Podesta and Spiro studied the existence of Kähler-Einstein metrics on homogeneous toric bundles [PS10]. These are homogeneous bundles over the flag manifold of a complex semisimple group, with fiber a toric variety. They studied these varieties to determine when they were Fano, and to determine when, in that case, the Futaki invariant vanished. Remark that these varieties are almost-homogeneous for a reductive group, and that the quotient of the open orbit under the maximal compact subgroup is again isomorphic to a $\mathbb{R}^r$ where here r is the dimension of the toric fiber. As in the toric case, one can use the compact group invariance to simplify the Kähler-Einstein equation, and one still obtains a real Monge-Ampère equation on $\mathbb{R}^r$ with the same type of prescribed asymptotic behavior as in the toric case. Applying the work of Wang and Zhu with minor modifications, Podesta and Spiro conclude that their manifolds admit a Kähler-Einstein metric if and only if the Futaki invariant vanishes.

To conclude this non-exhaustive list of examples, let us mention that Guan studied the existence of Kähler-Einstein (and more generally canonical) metrics on almost-homogeneous manifolds of cohomogeneity one, which means that the maximal compact subgroup acts with an hypersurface orbit. On these manifolds, which are hard to completely classify, the Kähler-Einstein equation must translate to some ordinary differential equation thanks to the compact group action. The latest articles by Guan on this subject are [Gua11a, Gua11b, Gua11c, Gua13].

2.1.4 Group compactifications

The Fano manifolds that we study in this thesis are some group compactifications. Let G be a connected reductive linear complex algebraic group (examples include $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{SL}_n(\mathbb{C})$, $(\mathbb{C}^*)^n$, etc.).

A compact toric variety Z of dimension r is a compactification of the torus $T = (\mathbb{C}^*)^r$. The fact that the torus acts on the whole manifold Z and that $T \subset Z$ is an orbit for the, say, left multiplication, means that Z is in fact a T-equivariant compactification of T. In other words, it is the data of an embedding j of T in a compact manifold Z such that Z admits a T action and j is T-equivariant as a map *i.e.* $j(t\cdot z) = t\cdot j(z)$. Equivalently, we could have used the action of T on itself by multiplication on the right by the inverse. Since T is abelian, both actions are equivalent.

When considering a non abelian reductive group G, the two actions are no longer equivalent. One could still consider left-G-equivariant compactifications, or right-G-equivariant compactifications, but we will consider the stronger assumption that the compactification is equivariant under both actions. We thus consider $G \times G$-equivariant compactifications of a complex connected reductive linear algebraic group G. For simplicity we will call these compactifications of G, or group compactifications when the group is not fixed.

One main reason to consider both actions is that such varieties are spherical. A variety X is called spherical if it is equipped with an action of a reductive group G such that a Borel subgroup B of G acts with an open orbit on X. It implies that G has an open and dense orbit in X, which is a homogeneous space under G. In particular, a spherical variety is almost homogeneous. They are the almost homogeneous varieties for which the classification theory is the most advanced and where many of the problems solved for toric varieties have a hope to find a similar resolution.

In fact, most of the examples of Fano Kähler-Einstein manifolds mentioned previously are spherical varieties. It is easy to see that compact manifolds homogeneous under a reductive group are spherical, and we have seen that toric varieties are spherical. The examples studied by Podesta and Spiro belong to the family of horospherical varieties, and contains both homogeneous manifolds and toric manifolds.

As we have seen, the toric varieties are also group compactifications, for abelian algebraic groups. On the opposite, the "least abelian" reductive groups are the semisimple groups. The most famous examples of compactifications of semisimple groups are the wonderful compactifications of adjoint semisimple groups constructed by De Concini and Procesi [DCP83]. In short, these are smooth and Fano manifolds, constructed as the closure of the $G \times G$-orbit of the class of the identity in $\mathbb{P}(\mathrm{End}(V_\lambda))$, where V_λ is the irreductible representation of G of highest weight a regular and dominant weight λ.

Donaldson suggested in his survey [Don08] to study the Kähler-Einstein existence problem, and more generally the existence of extremal or constant scalar curvature Kähler metrics, for spherical varieties. They fit in the first category of varieties presented by Donaldson in [Don08, Section 4] because they are multiplicity free. Remark that the smaller class of group compactifications that we consider here in fact also fits in the second category of manifolds presented by Donaldson in [Don08, Section 4].

2.2 Results and organization of the work

2.2.1 Chapter 3

The aim of Chapter 3 is to give the tools necessary to work on a reductive group G, and thus on the open and dense orbit of a compactification of G. The first section provides the definition of reductive groups and recalls many usual tools to study them. The root system Φ associated to such a group will be very important as our results will always be stated in terms of this root system, for the part coming from the big orbit.

The second section deals with the action of a maximal compact group K on G both on the left and on the right. For a reductive group, the corresponding quotient of G by $K \times K$ is a closed cone in $\mathbb{R}^r$ where r is the rank of the group. We identify it with the closed positive Weyl chamber $\overline{\mathfrak{a}^+}$ in the Cartan subalgebra $\mathfrak{a}$ of the Lie algebra of a maximal torus of G.

We consider functions on G invariant under $K \times K$. These correspond to functions on the quotient, and we recall how a result of Azad and Loeb allows to characterize plurisubharmonicity of a function on G as convexity of the function on the quotient. Then we recall the KAK integration formula, which translates the integral of a $K \times K$-invariant function on G with respect to a Haar measure as an integral on the quotient with respect to a measure on the quotient cone, absolutely continuous with respect to a Lebesgue measure, with an explicit potential denoted by J. Finally we compute the complex Hessian $\mathrm{Hess}_{\mathbb{C}}(\psi)$ of a $K \times K$-invariant function ψ on G in a choice of local coordinates.

Theorem 2.1. *Let ψ be a $K \times K$ invariant function on G, and f the associated function on $\mathfrak{a}$. Then in an appropriate choice of coordinates and for $a \in \mathfrak{a}^+$, the complex Hessian matrix of ψ is diagonal by blocks, equal to:*

$$\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(f)(a) & 0 & & 0 \\ 0 & M_{\alpha_1}(a) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_p}(a) \end{pmatrix}$$

where $\mathrm{Hess}_{\mathbb{R}}(f)$ is the real Hessian of f, the $(\alpha_i)_{i\in\{1,\dots,p\}}$ run over the positive roots of Φ and M_α is defined by:

$$M_\alpha(a) = \frac{1}{2}\alpha(\nabla f(a)) \begin{pmatrix} \coth(\alpha(a)) & i \\ -i & \coth(\alpha(a)) \end{pmatrix}$$

where ∇f is the gradient of f with respect to a fixed scalar product on $\mathfrak{a}$ extending the Killing form on the semisimple part.

As we have seen in this introduction, this computation will be very important to express the Kähler-Einstein equation on the big orbit of a group compactification. More precisely, we will use the determinant of this Hessian, the complex Monge-Ampère, denoted by $\mathrm{MA}_{\mathbb{C}}(\psi)$.

Corollary 2.2. *Let ψ be a $K\times K$ invariant function on G, and f the associated function on $\mathfrak{a}$. Then in an appropriate choice of coordinates and at $a \in \mathfrak{a}^+$, if r denotes the rank of G and p the number of positive roots, we have*

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{4^{r+p}}\mathrm{MA}_{\mathbb{R}}(f)(a) \prod_{\alpha\in\Phi^+} \frac{\alpha(\nabla f(a))^2}{\sinh^2(\alpha(a))}.$$

2.2.2 Chapter 4

Chapter 4 is an introduction to group compactifications with a focus on the polytope associated to a polarization of such a variety. The polytope contains all the information about the boundary divisor, and about the chosen ample line bundle. This is a generalization of the polytope associated to a polarized toric variety and in fact we use this usual correspondence to present the case of group compactifications. Indeed, given a compactification X of G, consider T a maximal torus in G and Z its closure in X. Then Z is a toric variety containing all the information about the compactification (if G is fixed). We give some examples of smooth and Fano group compactifications. There are many such examples: every adjoint semisimple group admits a canonical group compactification, called its wonderful compactification, and it is smooth and Fano.

After that, we consider hermitian metrics with non negative curvature on linearized ample line bundles on group compactifications. To such a metric h we associate a plurisubharmonic function on G. Provided h is $K\times K$-invariant, this determines, by Chapter 3, a convex function on $\mathfrak{a}^+$, called the convex potential of h. We prove that the fact that h is a metric on a polarization of X prescribes the asymptotic behavior of the convex function, and describe this asymptotic behavior in terms of the associated polytope.

Theorem 2.3. *Let (X, L) be a polarized compactification of G, with associated polytope P. The singular hermitian $K\times K$-invariant metrics h on L with non negative current curvature are in bijection with the convex W-invariant functions $\varphi : \mathfrak{a} \longrightarrow \mathbb{R}$ satisfying the condition that there exists a $C_1 \in \mathbb{R}$ such that*

$$\varphi(x) \le f_P(x) + C_1$$

on $\mathfrak{a}$ with f_P the support function of the polytope $2P$. Furthermore, h is locally bounded if and only if there exists in addition a constant C_2 such that

$$f_P(x) + C_2 \le \varphi(x) \le f_P(x) + C_1.$$

The function φ associated to h is its convex potential.

At the end of the chapter we combine all we have presented up to here (including the computation of the complex Hessian) to give a link between the degree of an ample line bundle L and the volume of its polytope with respect to the Duistermaat-Heckman measure.

Proposition 2.4. *Let (X, L) be a smooth polarized compactification of G, corresponding to the polytope P. Then*

$$\deg(L) = C \int_{2P^+} \prod_{\alpha \in \Phi^+} (\alpha(p))^2 dp$$

for some constant C depending only on the group G.

This result is consistent with the explicit formula obtained by Kazarnovskii [Kaz87] and Brion [Bri89], but we do not determine the constant C explicitly here, because we will not need it in the following.

2.2.3 Chapter 5

In Chapter 5 we compute the α-invariant of an ample line bundle on a Fano group compactification. More precisely we compute the α-invariant with respect to the action of a maximal compact subgroup $K \times K$ of $G \times G$.

We do this by computing the log canonical thresholds of any $K \times K$-invariant non negatively curved hermitian metric h on L in terms of a convex body associated to h. This convex body, that we call the Newton body of h, is the domain of the Legendre-Fenchel transform of the convex potential of h. The method to compute the log canonical threshold is to reduce to an integrability criterion on the open orbit G, then on $\mathbb{R}^r$ by the KAK integration formula. At this point we have to determine when the exponential of a concave function is integrable with respect to the measure with potential J against the Lebesgue measure. Using the integrability criterion derived by Guenancia as an analytic proof of Howald's computation of log canonical thresholds of monomial ideals, we can get such a criterion in terms of the Newton body.

To get an expression of the α-invariant, we show which are the metrics with potentially minimal log canonical threshold. The combination of these allows to prove the following theorem.

Theorem 2.5. *Let G be a connected complex reductive group, and X be a smooth Fano compactification of G. Let L be an ample $G \times G$-linearized line bundle on X, whose associated polytope is P. Denote by Q the polytope associated to the anticanonical line bundle $-K_X$. Then*

$$\alpha_{K \times K}(L) = \sup\{c > 0; c(P + (-P^W)) \subset Q \ominus H\},$$

where P^W denotes the subset of W-invariant points of P, H is the convex hull of the images by W of the sum of positive roots, and $\ominus$ is the Minkowski difference of convex sets.

Moreover, if the group G is semisimple, or if there are enough additional symmetries, there is a metric whose log canonical threshold is the α-invariant.

Corollary 2.6. *Assume that G is a semisimple group. Then*

$$\alpha_{K \times K}(L) = \sup\{c \geq 0; cP \subset Q \ominus H\}.$$

It is also the inradius of $Q \ominus H$ with respect to P.

Finally we compute the α-invariant for some examples of group compactifications.

Corollary 2.7. *Let X be the wonderful compactification of $\mathrm{PSL}_{n+1}(\mathbb{C})$, then*

$$\alpha_{K\times K}(-K_X) = \frac{1}{1+\lceil \frac{n}{2}\rceil(\lfloor \frac{n}{2}\rfloor+1)}.$$

From this example we see that such manifolds do not satisfy Tian's sufficient condition in general. In fact it is not hard to convince oneself that it will not be satisfied for most wonderful compactifications. Thus we need to use another method.

The formula we obtained applies also to toric manifolds and allows to recover our previous computation of the α-invariant on toric manifolds [Del15]. The text of this article is also reproduced in Appendix A.

2.2.4 Chapter 6

Chapter 6 contains the main result of the thesis. Namely we determine a necessary and sufficient condition for the existence of a Kähler-Einstein metric on a group compactification. This condition is further numerically computable in terms of the polytope of the group compactification.

Let G be a connected complex reductive group, X a smooth and Fano compactification of G, with associated polytope P. Let Φ be the root system of G, Φ^+ a system of positive roots. We denote by P^+ the intersection of P with the positive Weyl chamber, by 2ρ the sum of the positive roots, and by Ξ the relative interior of the closed cone generated by the simple roots. Finally, dp denotes the Lebesgue measure normalized by the lattice of characters.

Theorem 2.8. *There exists a Kähler-Einstein metric on the smooth and Fano group compactification X if and only if the barycenter*

$$\mathrm{bar}_{DH}(P^+) := \left(\int_{P^+} p \prod_{\alpha\in\Phi^+}(\alpha(p))^2 dp\right)\left(\int_{P^+}\prod_{\alpha\in\Phi^+}(\alpha(p))^2 dp\right)^{-1}$$

of P^+ with respect to the Duistermaat-Heckman measure is in $2\rho+\Xi$.

When G is semisimple, Ξ is the open cone generated by the simple roots of Φ. Remark that when G is not semisimple, the dimension of Ξ is strictly smaller than r. In particular, for G a torus, ρ is the origin and $\Xi = \{0\}$, so we recover the usual toric criterion. Indeed, the Duistermaat-Heckman measure then is just the Lebesgue measure on $P = P^+$, so the criterion is just that the barycenter of P is the origin.

We provide some examples of computations and obtain new examples of Kähler-Einstein Fano manifolds but also an example of compactification of a semisimple group that admits no Kähler-Einstein metrics.

Example 2.9. The wonderful compactifications of semisimple rank two complex groups admit Kähler-Einstein metrics.

These examples are new examples of Kähler-Einstein metrics. The knowledge of the connected component of their automorphism groups (see Chapter 4) shows that they are not homogeneous, and not toric.

The following non Kähler-Einstein example shows in particular that the vanishing of the Futaki invariant on a group compactification is not enough to ensure the existence of a Kähler-Einstein metric, whereas it was for the classes of spherical varieties previously studied (homogeneous, toric or horospherical).

Example 2.10. The blow up of the wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$ at the closed orbit is a smooth Fano group compactification which admits no Kähler-Einstein metrics.

To prove the theorem, the first step is to translate the Kähler-Einstein equation restricted to the dense orbit as a real Monge-Ampère equation on a cone in $\mathbb{R}^r$. Then we follow the same strategy as Wang and Zhu to get C^0 estimates. However, here the equation is different, the functions we study are different and one has to take extra care of what happens near the walls of the cone.

Furthermore, Wang and Zhu used the vanishing of the Futaki invariant to get C^0 estimates. In the case of a compactification of a semisimple group, the vanishing of the Futaki invariant does not give any information. To replace this we find a different integral invariant for Fano group compactifications, which is the barycenter of the polytope with respect to the Duistermaat-Heckman measure. The condition, as seen in the theorem, is no longer the vanishing of this integral invariant, but the fact that this barycenter is in a certain zone in the polytope. Equivalently, as the vanishing of the Futaki invariant is equivalent to the vanishing of a finite number of integrals, here we obtain a set of inequalities that must be satisfied by some integrals.

During the proof, we also compute the greatest Ricci lower bound when there are no Kähler-Einstein metrics. This is the maximal time of existence of a solution in the continuity method, which is how we compute it, but also the supremum of all $t < 1$ such that there exists a Kähler form ω in $c_1(X)$ with $\mathrm{Ric}(\omega) \geq t\omega$ [Szé11].

Theorem 2.11. *Assume there are no Kähler-Einstein metrics on X and let $R(X)$ be the greatest Ricci lower bound of X. Then*

$$R(X) = \sup\left\{t < 1; \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(P^+)) + 2\rho \in (P^+ + (-\Xi))\right\}.$$

Example 2.12. When X is the manifold from example 2.10, we compute the exact value of the greatest Ricci lower bound, which is

$$R(X) = \frac{1046175339}{1236719713} \simeq 0.8459\ldots$$

Chapter 3

Reductive groups and invariant functions

In this chapter we first introduce reductive groups and recall some results about reductive and semisimple groups that will be used in the rest of the text. A very important combinatorial data associated to such a group is its root system. Most of the first section of this chapter deals with the root system, the corresponding root decomposition of the Lie algebra of the group, and the classification of reductive groups that follows from it. This classification will allow to understand some examples of group compactifications that we will later consider.

The second section of this chapter deals with the action of the maximal compact subgroup of a reductive group by left and right multiplication. This action is extremely important in the setting of Kähler-Einstein metrics, and we use a classical reductive group decomposition to compute the complex Hessian and complex Monge-Ampère of a function on the group invariant under both actions of the compact group.

3.1 Reductive groups

3.1.1 Definition and maximal compact subgroup

There are several definitions for reductive groups, that are equivalent over the field $\mathbb{C}$. Let G be a complex connected linear algebraic group i.e. a connected algebraic subgroup of some $\mathrm{GL}_N(\mathbb{C})$. The first definition we give is in terms of the unipotent radical of the group G. Then we will give other characterizations that are equivalent. Let us first recall some usual notions in group theory. We use the books [Spr98, Bor91] as references for this section.

The *derived subgroup* $\mathcal{D}(G)$ of a group G is the subgroup generated by all commutators of elements of G. Given a group G, one can consider its *derived*

series
$$G \rhd \mathcal{D}(G) \rhd \mathcal{D}(\mathcal{D}(G)) \cdots$$
A group G is *solvable* if its derived series eventually reaches the trivial subgroup $\{e\}$ of G. The *radical* $R(G)$ of an algebraic group G is the maximal connected, normal and solvable subgroup of G.

An algebraic group G is said *unipotent* if it is isomorphic to a closed subgroup of some UT_n, where UT_n denotes the group of $n \times n$ upper triangular matrices with all diagonal coefficients equal to one. The *unipotent radical* $R_u(G)$ of an algebraic group G is the maximal connected, normal and unipotent subgroup of G.

This allows to define reductive and semisimple groups.

Definition 3.1. An algebraic group G is *reductive* if its unipotent radical $R_u(G)$ is trivial. It is *semisimple* if its radical $R(G)$ is trivial.

Example 3.2. A torus $(\mathbb{C}^*)^n$ is a reductive group that is not semisimple, and a unipotent group is not reductive.

Remark 3.3. Remark that $R_u(G) \subset R(G)$, so a semisimple group is reductive. In particular, all simple complex algebraic groups are reductive (and semisimple).

In fact we can obtain all reductive groups from torus and semisimple ones, as the following proposition shows.

Proposition 3.4. *[Spr98, 7.3.1 and 8.1.6] Let G be a reductive group, then $R(G)$ is the identity component of the center $Z(G)$ of G, and it is a torus, $\mathcal{D}(G)$ is a semisimple group, and G is the quotient of $\mathcal{D}(G) \times R(G)$ by a finite central subgroup.*

Another characterization, which explains the name, is in terms of representations.

Proposition 3.5. *A group G is reductive if and only if all rational representations of G are reducible, i.e. are direct sums of irreducible representations.*

The last characterization will be very useful for us. It gives a strong link between a reductive group G and a maximal compact subgroup K of G.

Proposition 3.6. *[see [Sch00], Chapter 5, and references therein] Let K be a maximal compact subgroup of G. Then G is reductive if and only if G is isomorphic to the* complexification *of K.*

Example 3.7. The general linear group $\mathrm{GL}_n(\mathbb{C})$ is a reductive group, it is the complexification of the unitary group $U(n)$.

3.1.2 Lie algebras and Killing form

Our reference book on Lie algebras is [Hum72]. From now on, G will denote a reductive group, and K a maximal compact subgroup of G. Let $\mathfrak{g}$, respectively $\mathfrak{k}$, denote the Lie algebras of G and K. Then $\mathfrak{g}$ is also the complexification of $\mathfrak{k}$:

$$\mathfrak{g} = \mathfrak{k} \otimes \mathbb{C}.$$

The induced complex conjugation on $\mathfrak{g}$ is a *Cartan involution* for G. When K is fixed we will denote this involution by θ.

A Lie algebra $(\mathfrak{g}, [\cdot,\cdot])$ acts on itself through the *adjoint* Lie algebra action: for $x, y \in \mathfrak{g}$, it is defined by $\mathrm{ad}(x)(y) := [x, y]$.

Definition 3.8. The *Killing form* B of $\mathfrak{g}$ is the bilinear symmetric form on $\mathfrak{g}$ defined for $x, y \in \mathfrak{g}$ by

$$B(x, y) = \mathrm{Tr}(\mathrm{ad}(x)\mathrm{ad}(y)).$$

Example 3.9. The Lie algebra of $\mathrm{SL}_n(\mathbb{C})$ is denoted by $\mathfrak{sl}_n(\mathbb{C})$ and consists of the matrices whose trace is zero. Let us describe the Killing form on $\mathfrak{sl}_2(\mathbb{C})$. A usual basis for $\mathfrak{sl}_2(\mathbb{C})$ is the basis (h_2, g_2, g_{-2}) where

$$h_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, g_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, g_{-2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

Then we can express the Killing form:

$$B(a_1h_2 + a_2g_2 + a_3g_{-2}, b_1h_2 + b_2g_2 + b_3g_{-2}) = 8a_1b_1 + 4(a_2b_3 + a_3b_2).$$

A *semisimple Lie algebra* is a Lie algebra that is a direct sum of simple Lie algebras. The Lie algebra of a semisimple Lie group is semisimple.

A fundamental result about the Killing form is that a Lie algebra $\mathfrak{g}$ is semisimple if and only if its Killing form is non degenerate. Furthermore it is then negative definite on the compact real form $\mathfrak{k}$ of $\mathfrak{g}$.

It is clear on the other hand that the Killing form on an abelian Lie algebra vanishes everywhere.

In the case of the Lie algebra $\mathfrak{g}$ of a reductive group, Proposition 3.4 shows that we can decompose the Lie algebra as $\mathfrak{g} = Z(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ where $[\mathfrak{g}, \mathfrak{g}]$ is the Lie algebra of $\mathcal{D}(G)$ and as such is semisimple, and $Z(\mathfrak{g})$ is abelian and is the center of $\mathfrak{g}$. The Killing form B on $\mathfrak{g}$ is nondegenerate on $[\mathfrak{g}, \mathfrak{g}]$ and zero on $Z(\mathfrak{g})$. We can also observe that these two subalgebras are orthogonal with respect to the Killing form.

We can choose any nondegenerate bilinear symmetric form on $Z(\mathfrak{g})$ to extend $B|_{[\mathfrak{g},\mathfrak{g}]}$ to a nondegenerate bilinear symmetric form on the whole of $\mathfrak{g}$. We fix such a choice in the following, and denote it by $\langle \cdot, \cdot \rangle$.

3.1.3 Roots

We begin by defining the root system associated to a semisimple Lie algebra, then relate this to semisimple groups. This will also be applied to reductive groups through Proposition 3.4.

Semisimple Lie algebras and root systems

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Recall that $\mathfrak{g}$ acts linearly on itself through the adjoint action. Choose $\mathfrak{t}$ a maximal abelian subalgebra of $\mathfrak{g}$ consisting of elements h such that $\mathrm{ad}(h)$ is diagonalisable. Restricted to $\mathfrak{t}$, which is abelian, the adjoint action is simultaneously diagonalisable and the corresponding eigenspace decomposition of $\mathfrak{g}$ is called the *root decomposition*. Using the notation

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; \mathrm{ad}(h)(x) = \alpha(h)x \ \forall h \in \mathfrak{t}\}$$

for any $\alpha \in \mathfrak{t}^*$, and denoting by Φ the set of nonzero $\alpha \in \mathfrak{t}^*$ such that $\mathfrak{g}_\alpha \neq \{0\}$, the root decomposition reads:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Furthermore, we have $\mathfrak{g}_0 = \mathfrak{t}$ [Hum72, 8.2], and any two Cartan subalgebras $\mathfrak{t}$ are conjugate by an automorphism of $\mathfrak{g}$.

The set Φ is called the *root sytem* of $\mathfrak{g}$, its elements the *roots* of $\mathfrak{g}$.

The Killing form B allows to associate to each root $\alpha \in \Phi$ the unique element h_α of $\mathfrak{t}$ such that $\alpha(h) = B(h_\alpha, h)$ for all $h \in \mathfrak{t}$. Let us introduce also the notation $(\alpha, \beta) := B(h_\alpha, h_\beta)$.

The root system Φ satisfies the following conditions:

- Φ spans a real subspace E of $\mathfrak{t}^*$, of real dimension equal to the complex dimension of $\mathfrak{t}$, on which $(\cdot,\cdot)$ extends to a positive definite form;
- if $\alpha \in \Phi$, there are exactly two multiples of α in Φ which are α and $-\alpha$;
- if $\alpha, \beta \in \Phi$, then the image $\beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha$ of β by the reflection determined by α is in Φ;
- if $\alpha, \beta \in \Phi$, then $\frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$.

In fact, these conditions are the axioms defining an *abstract* (reduced) *root systems* [Hum72, 9.2], and the following classical theorem states that complex semisimple Lie algebras are classified by root systems.

Theorem 3.10. *[Hum72, 18.4] For any abstract root system Φ, there exists a unique (up to isomorphism) complex semisimple Lie algebra $\mathfrak{g}$ whose root system is Φ.*

In addition, the root systems of simple Lie algebras are combinatorially classified, and the root system of a direct sum is the product of the root systems, so all semisimple complex Lie algebras are combinatorially classified in this way.

Let us also record some properties of the root decomposition.

Proposition 3.11. *[Hum72, 8.3 and 8.4] Let $\alpha, \beta \in \Phi$, then*

- *the root space $\mathfrak{g}_\alpha$ is of complex dimension one;*
- *if $\alpha + \beta \in \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$;*
- *the subspace $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is in $\mathfrak{t}$ and one dimensional;*
- *more precisely, if $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$, then $[x, y] = B(x, y)h_\alpha$.*

More generally, if $\alpha, \beta \in \mathfrak{a}^*$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, so if $0 \neq \alpha + \beta \notin \Phi$ then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \{0\}$.

To a semisimple group G one can associate the root system Φ of its Lie algebra $\mathfrak{g}$. However, two non isomorphic semisimple groups can have the same Lie algebra, for example this is the case with SL_n and PGL_n. To distinguish two such groups one needs extra data. This will be discussed in the next section.

Root system of a reductive group

Let now G be reductive, with maximal compact group K. Choose S a maximal torus of K, and let T be its complexification in G. Then T is also a maximal torus of G.

Definition 3.12. The complex dimension $rk(G) := r$ of T, which is also the real dimension of S, is called the *rank* of G.

Let Φ be the root system of (G, T), i.e. the root system of the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$ with the choice of maximal abelian subalgebra $\mathfrak{t}_{ss}$ the semisimple part of the Lie algebra of T. The Cartan involution θ induces an involution on $\mathfrak{t}_{ss}^*$ which preserves the roots, and we still denote by θ the corresponding involution of Φ. Furthermore, θ sends $\alpha \in \Phi$ to $-\alpha$.

We will denote by $\mathfrak{a}$ the vector space $i\mathfrak{s} \subset \mathfrak{t}$ where $\mathfrak{s} = \mathfrak{t} \cap \mathfrak{k}$ is the Lie algebra of S. We denote by A the image of $\mathfrak{a}$ in G by the exponential map.

3.1.4 Weights

Semisimple case

Let Φ be the root system of a semisimple Lie algebra $\mathfrak{g}$, let E be the subspace of $\mathfrak{t}^*$ generated by the roots of Φ. Define M_{sc} to be the set of all $m \in E$ such that $\frac{2(m,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ for all $\alpha \in \Phi$, and call its elements the *weights* of Φ. The set M_{sc} is a lattice in E.

A *Weyl wall* in E is a hyperplane defined by an equation of the form $(x, \alpha) = 0$ for some root $\alpha \in \Phi$. Call a weight m *regular* if $(m, \alpha) \neq 0$ for all $\alpha \in \Phi$, *i.e.* m is not on any Weyl wall.

A subset Δ of Φ is a set of *simple roots* if it is a basis of E, and any root in Φ has either all of its coordinates positive in this basis or all of its coordinates negative. The set of roots with positive coordinates is then denoted by Φ^+ and its elements are called the *positive roots* of Φ. Choose a set of simple roots Δ in Φ. A weight m is called *dominant* if it satisfies $(m, \alpha) \geq 0$ for all $\alpha \in \Delta$. This is equivalent to $(m, \alpha) \geq 0$ for all $\alpha \in \Phi^+$.

The basis of E formed by the m_i such that $\frac{2(m_i,\alpha_j)}{(\alpha_j,\alpha_j)} = \delta_{ij}$ is also a basis of the lattice M_{sc}. Its elements are called the *fundamental weights* of Φ, and the dominant weights are the elements with positive coordinates in this basis. The closed cone generated by the fundamental weights

$$\overline{E^+} := \{\sum_{i=1}^{r} x_i m_i ; x_i \geq 0\}$$

is called the *positive closed Weyl chamber* of E. We will say that its interior E^+ is the (open) *positive Weyl chamber* of E.

The positive Weyl chamber is a connected component of the complement of the union of Weyl walls. The other components are called the *Weyl chambers* of E. Each would be the positive Weyl chamber for an appropriate choice of simple roots. In fact the *Weyl group* W of Φ, defined as the finite group generated by the reflections $m \mapsto m - \frac{2(m,\alpha)}{(\alpha,\alpha)}\alpha$ for $\alpha \in \Phi$, acts transitively on the set of Weyl chambers, and the closed positive Weyl chamber is a fundamental domain for the action of W on E.

Let G be a semisimple group and T be a maximal torus in G. Define the lattice of *weights* M of G as the lattice of characters of T. We will see that it allows to distinguish the groups with the same Lie algebra.

Given a root system Φ, let M_{ad} be the lattice generated by the roots of Φ, and recall that M_{sc} the lattice generated by the fundamental weights of Φ.

Theorem 3.13. *[FH91, Theorem 23.16] Given a root system Φ and a lattice M between M_{ad} and M_{sc}, there exists a unique (up to isomorphism) semisimple group G whose root system is Φ and whose lattice of weights is M.*

The semisimple group G_{ad} with weight lattice M_{ad} is called *adjoint*, and the semisimple group G_{sc} with weight lattice M_{sc} is called *simply connected*. In fact if $M_1 \subset M_2$ are the weight lattices of two semisimple groups G_1 and G_2 with the same root system then G_1 is isomorphic to the quotient of G_2 by a finite group.

Reductive case

Let us now consider a reductive group G. We can still define the lattice of weights M of G by choosing a maximal torus T and considering its lattice of characters. Furthermore we associate to G the root system Φ of its derived subgroup $D(G)$ as before. Then the classification of reductive groups follows from Proposition 3.4 and Theorem 3.13. Remark also that if $\mathfrak{t}$ denotes the Lie algebra of the chosen maximal torus, we still have the root decomposition with the properties of Proposition 3.11:

$$\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha.$$

Let also N be the lattice of complex one parameter subgroups of T. We may also call N the lattice of *coweights* of G. Both M and N are free abelian groups of rank r naturally dual to each other.

Identify $\mathfrak{a}$ with $N \otimes \mathbb{R}$ and $\mathfrak{a}^*$ with $M \otimes \mathbb{R}$. Remark that if G is semisimple, $\mathfrak{a}^*$ is exactly E.

Choose a set of positive roots Φ^+. This allows to define a closed *Weyl chamber* $\overline{\mathfrak{a}^+}$ in $\mathfrak{a}$, by $x \in \overline{\mathfrak{a}^+}$ if and only if $\alpha(x) \geq 0$ for all $\alpha \in \Phi^+$. Denote by A^+ the subset of A which is the image by the exponential map of the closed Weyl chamber $\overline{\mathfrak{a}^+}$. The *open Weyl chamber* $\mathfrak{a}^+$ is defined as the interior of the closed Weyl chamber. In the case when G is semisimple this is equivalently defined by $x \in \mathfrak{a}^+$ if and only if $\alpha(x) > 0$, but not in the case of reductive groups.

The finite group $W = N_G(T)/T$ is called the *Weyl group* of G with respect to the maximal torus T. It is also the Weyl group of the root system Φ. It acts on T and induces an action on $\mathfrak{a}$. The *closed* Weyl chamber $\overline{\mathfrak{a}^+}$ is a fundamental domain for the action of W on $\mathfrak{a}$.

3.1.5 Examples

Rank one

There are only two semisimple groups of rank one. They are $\mathrm{PGL}_2(\mathbb{C})$ which is adjoint and $\mathrm{SL}_2(\mathbb{C})$ which is simply connected. Their common Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ which has only two roots in its root system. So there is only one root system of rank one that is denoted by A_1.

Identifying one root with $2 \in \mathbb{R}$ gives an identification of $\mathfrak{a}^*$ with $\mathbb{R}$. If we choose this root as the positive root then the positive Weyl chamber is $\mathbb{R}^+$. The root lattice, generated by the roots, is $2\mathbb{Z}$ and the weight lattice of $\mathfrak{sl}_2(\mathbb{C})$ is $\mathbb{Z}$.

The weight lattice of $\mathrm{PGL}_2(\mathbb{C})$ is thus $2\mathbb{Z}$ and the weight lattice of $\mathrm{SL}_2(\mathbb{C})$ is $\mathbb{Z}$. Identifying $\mathfrak{a}$ with its dual we get also that the coweight lattice of $\mathrm{PGL}_2(\mathbb{C})$ is $\mathbb{Z}$ and the coweight lattice of $\mathrm{SL}_2(\mathbb{C})$ is $2\mathbb{Z}$.

Rank two

There are four root systems of rank 2, up to isomorphism. One is obtained as a product of two copies of A_1. This is for example the root system of $\mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})$. The other three are irreducible. They are denoted by A_2, B_2 and G_2.

Figures 3.1 ,3.2 and 3.3 give the representations of these root systems in the usual euclidean plane $\mathbb{R}^2$. The black circles represent the roots, the white circles represent the rest of the root lattice and the crosses represent the points of the weight lattice that are not in the root lattice. From this and Theorem 3.13 we see that there are two semisimple groups with root system A_2, the adjoint one which is $\mathrm{PGL}_3(\mathbb{C})$ and the simply connected one which is $\mathrm{SL}_3(\mathbb{C})$. For type B_2 there is again one adjoint group $\mathrm{SO}_5(\mathbb{C})$ and one simply connected group $\mathrm{Sp}_4(\mathbb{C})$. Finally for type G_2 there is only one group, denoted again G_2 that is both adjoint and simply connected.

On the figures are also represented the Weyl chambers, which coincide for $\mathfrak{a}$ and $\mathfrak{a}^*$ in the identifications we made.

3.1.6 A basis of $\mathfrak{g}$

We will now fix a connected reductive group G and give a basis of $\mathfrak{g}$, taking into account the fact that G is the complexification of a compact group K. This basis will give rise to local complex coordinates on G in which we will be able to compute the complex Hessian matrix of a $K \times K$-invariant function on G.

Recall that we have two decompositions of $\mathfrak{g}$:

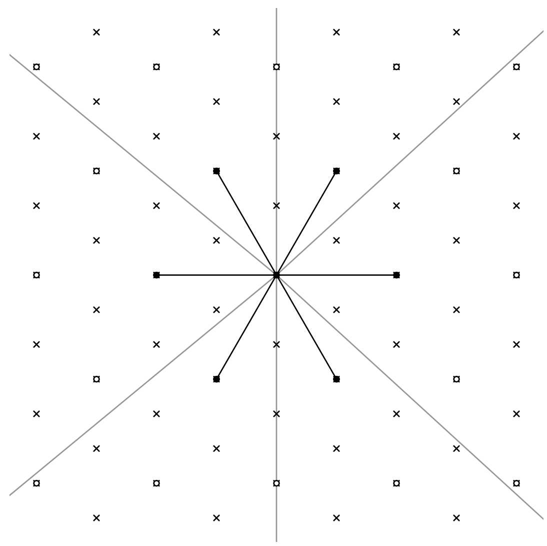


Figure 3.1: Root system A_2

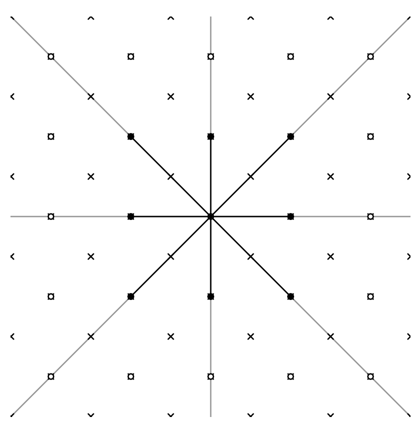


Figure 3.2: Root system B_2

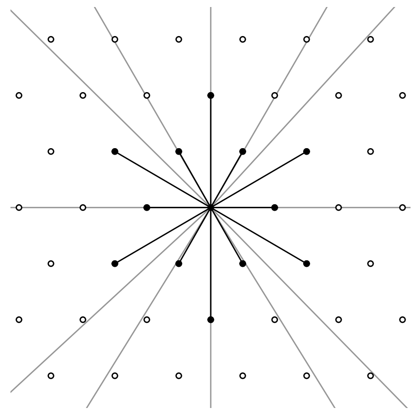


Figure 3.3: Root system G_2

– the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus i\mathfrak{k}$,
– and the root decomposition $\mathfrak{g} = \mathfrak{t} \oplus \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$.

We want to combine these two to obtain a basis suitable for computations.

Define the following real subspaces of $\mathfrak{k}$:

$$\mathfrak{k}_\alpha := \{x \in \mathfrak{k}; \mathrm{ad}(h)^2(x) = \alpha(h)^2 x \ \forall h \in \mathfrak{g}_0\},$$

for $\alpha \in \Phi$. Remark that $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$.

It is linked with the root decomposition by the equality:

$$\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha} = \mathfrak{k}_\alpha \oplus i\mathfrak{k}_\alpha.$$

Remark that if θ denotes the Cartan involution then $\theta(\mathfrak{g}_\alpha) = \mathfrak{g}_{-\alpha}$. Each $\mathfrak{g}_\alpha$ is of complex dimension one, so each $\mathfrak{k}_\alpha$ is of real dimension two.

This gives a decomposition of $\mathfrak{k}$:

$$\mathfrak{k} = \mathfrak{s} \oplus \bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha.$$

Suppose for the moment that $\mathfrak{g}$ is semisimple.There is a special complex basis of $\mathfrak{g}$ adapted to the root decomposition [Hel78, Chapter VI, Lemma 3.1]. Let $h_\alpha \in \mathfrak{a} = i\mathfrak{s}$ be the unique element such that $B(h, h_\alpha) = \alpha(h)$ for all $h \in \mathfrak{t}$. Remark that $h_{-\alpha} = -h_\alpha$. Helgason proves that we can choose generating vectors e_α of $\mathfrak{g}_\alpha$ such that

– e_α, $e_{-\alpha}$ and $[e_\alpha, e_{-\alpha}] = \frac{2}{\alpha(h_\alpha)} h_\alpha$ generate an $\mathfrak{sl}_2(\mathbb{C})$ subalgebra,
– $e_{-\alpha} = -\theta(e_\alpha)$.

We just scale the vectors e_α by $\sqrt{\frac{2}{\alpha(h_\alpha)}}$ to simplify the expressions, and thus we have a basis e_α of $\mathfrak{g}_\alpha$ such that $e_{-\alpha} = -\theta(e_\alpha)$ and $[e_\alpha, e_{-\alpha}] = h_\alpha$.

Now $\mathfrak{k}_\alpha$ admits $e_\alpha + \theta(e_\alpha)$, $ie_\alpha - i\theta(e_\alpha)$ as a real basis. This basis can also be written $e_\alpha - e_{-\alpha}$, $ie_\alpha + ie_{-\alpha}$. We complete this with any basis of $\mathfrak{s}$ and we get a real basis of $i\mathfrak{k}$, thus also a complex basis of $\mathfrak{g}$.

We can apply this in the reductive case also by considering the semisimple part $[\mathfrak{g}, \mathfrak{g}]$, then completing the basis by choosing any real basis of $Z(\mathfrak{g}) \cap \mathfrak{k}$.

Example 3.14. In the case of $\mathfrak{sl}_2(\mathbb{C})$, we described the Killing form previously, and the Cartan involution sends a matrix to the opposite of the conjugate of its transpose, so we get

$$h_2 = \begin{pmatrix} 1/4 & 0 \\ 0 & -1/4 \end{pmatrix}, e_2 = \begin{pmatrix} 0 & 1/2 \\ 0 & 0 \end{pmatrix}, e_{-2} = \begin{pmatrix} 0 & 0 \\ 1/2 & 0 \end{pmatrix},$$

and so the basis of $i\mathfrak{su}_2$ obtained is formed by the matrices

$$e_2 - e_{-2} = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, ie_2 + ie_{-2} = \begin{pmatrix} 0 & i/2 \\ i/2 & 0 \end{pmatrix}.$$

3.2 $K \times K$-invariant functions on G

In this section we will first recall the classical KAK decomposition of a reductive group. This is the decomposition that encodes the $K \times K$ orbits of G, or equivalently that describes the quotient under this action. It will be used in the following to manipulate $K \times K$-invariant functions on G. We recall or prove here the main tools for this.

The first is the translation of smoothness or positivity properties, the second is an integration formula adapted to the KAK decomposition, and the last, which is proved here, is a computation of the complex Hessian of a $K \times K$-invariant function in the basis defined in Section 3.1.6. This gives also an expression of the complex Monge-Ampère, which is what we will mainly use in the next chapters.

3.2.1 KAK decomposition and invariant functions

The KAK decomposition can be stated as the following decomposition of an element of G.

Proposition 3.15. *[Kna02, Theorem 7.39] Let $g \in G$ be any element, then we can write $g = k_1 t k_2$, with $k_1, k_2 \in K$, and $t \in A$. Furthermore, in this decomposition, t is uniquely determined up to the action of the Weyl group W.*

Another way to state this result is by saying that any $g \in G$ can be written $g = k_1 \exp(a) k_2$, where $k_1, k_2 \in K$ and $a \in \overline{\mathfrak{a}^+}$ uniquely determined by g. Indeed, we have seen before that $\overline{\mathfrak{a}^+}$ is a fundamental domain for the action of W on $\mathfrak{a}$. It means also that the quotient of G by the action of both left and right K action can be identified with A^+, or $\overline{\mathfrak{a}^+}$.

The KAK decomposition implies that a $K \times K$ invariant function ψ on G only depends on its values at points in A. Equivalently, one can consider the function f defined on $\mathfrak{a}$ by $f(a) = \psi(\exp(a))$. Furthermore, the function f is W-invariant.

We record here how some properties on ψ translate to properties on f. The first one is about smoothness.

Proposition 3.16. *[FJ78, Theorem 4.1] The correspondence $\psi \mapsto f$ gives a bijection between $K \times K$-invariant functions on G (resp. smooth $K \times K$-invariant functions on G) and W-invariant functions on $\mathfrak{a}$ (resp. smooth W-invariant functions on $\mathfrak{a}$).*

Next, the following result of Azad and Loeb tells us that plurisubharmonicity translates to convexity. This is to be related to the case of functions on $(\mathbb{C}^*)^n$ invariant under $(S^1)^n$, which is heavily used in toric geometry. In that case, a psh function on $(\mathbb{C}^*)^n$ corresponds to a convex function on $\mathbb{R}^n$. In fact, this easy result is a subcase of Azad and Loeb's result, with $G = (\mathbb{C}^*)^n$.

Indeed, for the reductive group $(\mathbb{C}^*)^n$, the compact torus $(\mathbb{S}^1)^n$ is a maximal compact subgroup, and the KAK decomposition reads

$$(\mathbb{C}^*)^n = (\mathbb{S}^1)^n (\mathbb{R}_+^*)^n (\mathbb{S}^1)^n.$$

However, since it is abelian, only one $(\mathbb{S}^1)^n$ factors counts, and the decomposition is given by taking on each factor the angle and the modulus. The set $A \subset G$ is thus $(\mathbb{R}_+^*)^n$ in this case and it is the image by the exponential of $\mathbb{R}^n$.

A smooth function ψ from an open subset of $\mathbb{C}^n$ to $\mathbb{R}$ is *plurisubharmonic* (we will say *psh*) if its complex Hessian is non negative. It is *strictly psh* if its complex Hessian is positive. This is still defined for, say, locally integrable functions, in the sense of distributions.

Proposition 3.17. *[AL92, Theorem 1] The correspondence $\psi \mapsto f$ gives a bijection between $K \times K$-invariant psh functions on G (resp. smooth strictly psh $K \times K$-invariant functions on G) and W-invariant convex functions on $\mathfrak{a}$ (resp. smooth strictly convex W-invariant functions on $\mathfrak{a}$).*

3.2.2 Haar measure and KAK integration formula

Let G be a reductive group. Then there exists a unique left-invariant positive smooth measure on G, up to a positive constant. Such a measure is called a *Haar measure*, and will be denoted by dg.

We restrict here to reductive groups but of course Haar measures exist on more general groups. Haar measures on reductive groups satisfy a stronger property: they are also invariant under the right action of G (we say that a reductive group is *unimodular*).

We will integrate $K \times K$-invariant functions on G with respect to the Haar measure, and we want to express such an integral in terms of the restriction to A of the function. Using the KAK decomposition, it is possible to make a variable change and get the formula in the following theorem, by computing a Jacobian. This computation was originally done by Harish-Chandra and can be found in a book of Helgason.

Let J denote the function on $\mathfrak{a}$ defined by

$$J(x) = \prod_{\alpha \in \Phi^+} \sinh^2(\alpha(x)).$$

Theorem 3.18. *[Kna02, Proposition 5.28] (see also [Hel84, Theorem 5.8]) Let dg denotes a Haar measure on G, and dx a Lebesgue measure on $\mathfrak{a}^+$, then there exists a constant $C > 0$ such that for all $K \times K$-invariant positive function ψ on G,*

$$\int_G \psi(g)dg = C \int_{\mathfrak{a}^+} J(x)\psi(\exp(x))dx.$$

It is easy to find a Haar measure on a Lie group. First choose any basis of the cotangent space at the neutral element e. Say here we choose a complex basis $dz_1, \ldots, dz_n$ of T_e^*G, and build the top exterior product

$$i^n dz_1 \wedge d\overline{z_1} \wedge \cdots \wedge dz_n \wedge d\overline{z_n}.$$

We multiplied by i^n to get a real form. Then transport this $2n$-form by the action of G on itself by right translation. This gives a smooth volume form on G invariant under the right action, so a Haar measure on G. Unimodularity implies that the volume form built this way is also $G \times G$-invariant.

3.2.3 Complex Hessian matrix on G

We will compute in this section the complex Hessian of a $K \times K$-invariant function on G in terms of the real Monge-Ampère of the associated W-invariant function on $\mathfrak{a}$. This computation is valid in an appropriate choice of complex coordinates on G: at the identity element $e \in G$, choose the basis of $T_eG = \mathfrak{g}$ given in Section 3.1.6. This gives a complex basis of T_eG, and by the action of G by multiplication on the right, we can transport this to a complex basis of any T_gG.

These also define local complex coordinates near every element of G. Indeed, the exponential being a biholomorphism from a neighborhood of $0 \in \mathfrak{g}$ to a neighborhood of the neutral element e in G, we get holomorphic coordinates near e. Then, composing with the multiplication on the right by $g \in G$, this defines holomorphic coordinates on a neighborhood of g. More precisely, if $(l_j)_{j=1}^n$ denotes the chosen basis of $\mathfrak{g}$, the map corresponding to the local coordinates is the map $\mathbb{C}^n \to G$ defined by

$$(z_1, \ldots, z_n) \mapsto \exp(z_1 l_1 + \cdots z_n l_n)g.$$

We will compute the complex Hessian with respect to these coordinates. If ψ is a function on G we denote by $\mathrm{Hess}_{\mathbb{C}}(\psi)(g)$ the complex Hessian and by $\mathrm{MA}_{\mathbb{C}}(\psi)(g)$ the determinant of the complex Hessian of ψ at g, called the complex Monge-Ampère, everything with respect to the coordinates given above.

If f is a function on $\mathfrak{a}$, then we denote by $\mathrm{MA}_{\mathbb{R}}(f)(x)$ the determinant of its real Hessian at x. We denote by ∇f the gradient of f with respect to the scalar product $\langle \cdot, \cdot \rangle$ on $\mathfrak{a}$.

Theorem 3.19. *Let ψ be a $K \times K$ invariant function on G, and f the associated function on $\mathfrak{a}$. Then in the coordinates above and for $a \in \mathfrak{a}^+$, the complex*

Hessian matrix of ψ is diagonal by blocks, equal to:

$$\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \begin{pmatrix} \frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(f)(a) & 0 & & 0 \\ 0 & M_{\alpha_1}(a) & & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & & M_{\alpha_p}(a) \end{pmatrix}$$

where the $(\alpha_i)_{i\in\{1,\dots,p\}}$ run over the positive roots of Φ and M_α is defined by:

$$M_\alpha(a) = \frac{1}{2}\alpha(\nabla f(a)) \begin{pmatrix} \coth(\alpha(a)) & i \\ -i & \coth(\alpha(a)) \end{pmatrix}.$$

Corollary 3.20. *Let ψ be a $K\times K$ invariant function on G, and f the associated function on $\mathfrak{a}$. Then in the coordinates above and at $a \in \mathfrak{a}^+$, if r denotes the rank of G and p the number of positive roots, we have*

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{4^{r+p}}\mathrm{MA}_{\mathbb{R}}(f)(a) \prod_{\alpha\in\Phi^+} \frac{\alpha(\nabla f(a))^2}{\sinh^2(\alpha(a))}.$$

Proof. Since $\mathrm{MA}_{\mathbb{R}}(f)(a) = \det(\mathrm{Hess}_{\mathbb{R}}(f)(a))$, we just have to compute the determinant of M_α. This is

$$\begin{aligned} \det(M_\alpha) &= (\frac{1}{2}\alpha(\nabla f(a)))^2(\coth(\alpha(a))^2 - 1) \\ &= (\frac{1}{2}\alpha(\nabla f(a)))^2\frac{\cosh(\alpha(a))^2 - \sinh(\alpha(a))^2}{\sinh(\alpha(a))^2} \\ &= (\frac{1}{2}\alpha(\nabla f(a)))^2\frac{1}{\sinh(\alpha(a))^2} \end{aligned}$$

□

Remark 3.21. Another way to write this is as

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{4^{r+p}}\mathrm{MA}_{\mathbb{R}}(f)(a)\frac{1}{J(a)} \prod_{\alpha\in\Phi^+} \alpha(\nabla f(a))^2$$

where J is the function involved in the KAK integration formula.

Example 3.22. Consider the case $G = \mathrm{PSL}_2(\mathbb{C})$. Then $\mathfrak{a}^+ \simeq \mathbb{R}_+^*$, and there is only one positive root that we can identify with the identity on $\mathbb{R}$. Then

$$\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{2}\begin{pmatrix} f''(a)/2 & 0 & 0 \\ 0 & f'(a)\coth(a) & if'(a) \\ 0 & -if'(a) & f'(a)\coth(a) \end{pmatrix}$$

and the complex Monge-Ampère reads:

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \frac{1}{16}f''(a)(f'(a))^2\frac{1}{\sinh^2(a)}.$$

Remark 3.23. Renormalizing correctly the basis we can, and will, assume that:

$$\mathrm{MA}_{\mathbb{C}}(\psi)(\exp(a)) = \mathrm{MA}_{\mathbb{R}}(f)(a)\frac{1}{J(a)}\prod_{\alpha\in\Phi^+}\alpha(\nabla f(a))^2.$$

Corollary 3.24. *By taking, this time, the trace of the complex Hessian, we recover the expression of the laplacian applied to a $K\times K$-invariant function on G, also called the* radial laplacian*:*

$$\begin{aligned}\Delta_r(\psi)(a) :=&\mathrm{Tr}(\mathrm{Hess}_{\mathbb{C}}(\psi)(\exp(a)))\\ =&\frac{1}{4}\mathrm{Tr}(\mathrm{Hess}_{\mathbb{R}}(f)(a)) + \sum_{\alpha\in\Phi^+}\alpha(\nabla f(a))\mathrm{coth}(\alpha(a)).\end{aligned}$$

The rest of the section is devoted to the proof of the theorem. The technique of the proof is based on the work of Bielawski [Bie04]. In particular, the idea to use the decomposition in Lemma 3.26 and the Baker-Campbell-Hausdorff formula appears in this article.

We begin by introducing these two tools.

The Baker-Campbell-Hausdorff formula

As a formal series in the variables x and y, the logarithm of $\exp(x)\exp(y)$ is well defined. We denote this by $\mathrm{BCH}(x,y)$. The Baker-Campbell-Hausdorff formula is the following.

Proposition 3.25. *There exists a neighborhood U of 0 in $\mathfrak{g}$ such that for all x and y in U, $\mathrm{BCH}(x,y)$ is convergent and defines an element of $\mathfrak{g}$, and we have*

$$\exp(x)\exp(y) = \exp(\mathrm{BCH}(x,y)).$$

Furthermore we know explicitly the first terms of $\mathrm{BCH}(x,y)$. We will only use the following:

$$\mathrm{BCH}(x,y) = x+y+\frac{1}{2}[x,y]+O$$

where O denotes terms of order higher than 2 in x and y.

A decomposition in $\mathfrak{g}$

Let $a\in\mathfrak{a}^+$. Let $\mathrm{Exp}(\mathrm{ad}(a))$ be the linear application $\mathfrak{g}\to\mathfrak{g}$ defined by

$$\mathrm{Exp}(\mathrm{ad}(a))(x) = \sum_{n=0}^{\infty}\frac{\mathrm{ad}(a)^n(x)}{n!}.$$

Recall that G acts on $\mathfrak{g}$ through the adjoint action Ad, and that we have the general relation

$$\mathrm{Exp}(\mathrm{ad}(a))(x) = \mathrm{Ad}(\exp(a))(x)$$

for $x\in\mathfrak{g}$.

Lemma 3.26. *Let $l \in \mathfrak{g}$ and $a \in \mathfrak{a}^+$. Then*

- *there exists $A \in \mathfrak{k}$, $B \in \mathfrak{a}$ and $C \in \mathrm{Ad}(\exp(a))(\bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha)$ such that*
$$l = A + B + C;$$
- *if $l \in \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$ then $B = 0$.*
- *if $l \in \mathfrak{k}_\alpha$, and l' denotes $\frac{1}{\alpha(a)}\mathrm{ad}(a)(il)$, then $l' \in \mathfrak{k}_\alpha$ and the decomposition above for il reads*
$$il = -\cosh(\alpha(a))l' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l');$$
- *if $l = e_\alpha + \theta(e_\alpha)$ then $l' = ie_\alpha - i\theta(e_\alpha)$;*
- *if $l = ie_\alpha - i\theta(e_\alpha)$ then $l' = -e_\alpha - \theta(e_\alpha)$.*

In the statement, the result is more and more precise as we know more precisely the element considered. In the proof we will begin by the very precise case of the elements of the basis and work our way up by linearity.

Proof. Let $a \in \mathfrak{a}^+$. We begin by the two last points. By definition of l', we have, if $l = e_\alpha + \theta(e_\alpha)$,

$$\begin{aligned} l' &= \mathrm{ad}(a)(il)/\alpha(a) \\ &= \mathrm{ad}(a)(ie_\alpha)/\alpha(a) + \mathrm{ad}(a)(i\theta(e_\alpha))/\alpha(a) \\ &= ie_\alpha - i\theta(e_\alpha) \end{aligned}$$

and if $l = ie_\alpha - i\theta(e_\alpha)$,

$$\begin{aligned} l' &= \mathrm{ad}(a)(il)/\alpha(a) \\ &= \mathrm{ad}(a)(-e_\alpha)/\alpha(a) + \mathrm{ad}(a)(\theta(e_\alpha))/\alpha(a) \\ &= -e_\alpha - \theta(e_\alpha). \end{aligned}$$

In particular, in both cases, l' is in $\mathfrak{k}_\alpha$. By linearity this is also true of l' for any $l \in \mathfrak{k}_\alpha$.

To prove the decomposition in the third point, it suffices to compute that, using the definition of $\mathfrak{k}_\alpha$,

$$\begin{aligned} \mathrm{Exp}(\mathrm{ad}(a))(l') &= \cosh(\alpha(a))l' + \sinh(\alpha(a))il \\ &= \mathrm{Ad}(\exp(a))(x). \end{aligned}$$

Then by linearity the first point holds true for any $l \in i\bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha$, with $B = 0$. But we have

$$\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha = \bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha \oplus i \bigoplus_{\alpha\in\Phi^+} \mathfrak{k}_\alpha,$$

so we have the decomposition for any $l \in \bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha$, with $B = 0$.

Finally for $l \in \mathfrak{t}$, it suffices to decompose l along $\mathfrak{t} = \mathfrak{s} \oplus \mathfrak{a}$. By linearity and the root decomposition, we obtain the proposition for any $l \in \mathfrak{g}$. $\square$

Using the Baker-Campbell-Hausdorff formula

We want to compute the complex Hessian of ψ in the chosen system of coordinates, at a point $\exp(a)$ for a in the open Weyl chamber $\mathfrak{a}^+$. If l_1 and l_2 are two vectors in the chosen basis of $\mathfrak{k}$, we thus want to compute:

$$H_{l_1,l_2}(a) := \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_{z_1,z_2=0} \psi(\exp(z_1 l_1 + z_2 l_2)\exp(a)).$$

There are different cases, according to the subspaces where l_1 and l_2 lie. We will first describe the part of the argument that is used in all cases, which relies on the Baker-Campbell-Hausdorff formula, and then deal with each case separately.

Using the decomposition from Lemma 3.26 on $z_1 l_1 + z_2 l_2$ we can write

$$z_1 l_1 + z_2 l_2 = A_1 + B_1 + C_1$$

with $A_1 \in \mathfrak{k}$, $B_1 \in \mathfrak{a}$ and $C_1 \in \mathrm{Ad}(\exp(a))(\mathfrak{k})$, and all are of homogeneous degree one in z_1 and z_2. Let

$$D_1 = \frac{1}{2}([B_1, A_1] + [C_1, A_1] + [C_1, B_1]),$$

it is of order two in z_1 and z_2.

Let us now use again Lemma 3.26 to get

$$D_1 = A_2 + B_2 + C_2.$$

with $A_2 \in \mathfrak{k}$, $B_2 \in \mathfrak{a}$ and $C_2 \in \mathrm{Ad}(\exp(a))(\mathfrak{k})$, and all are of homogeneous degree two in z_1 and z_2.

Then the Baker-Campbell-Hausdorff formula allows to prove the following lemma. This can be seen as an explicit infinitesimal KAK decomposition. Note that to lighten the notations we do not write the dependence on z_1, z_2, but all the terms defined above A_j, B_j, C_j and D_1 are in fact functions of these two complex variables.

Lemma 3.27. *We can write*

$$\exp(z_1 l_1 + z_2 l_2)\exp(a) = k_1 \exp(B_1 + B_2 + a + O) k_2$$

where O denotes terms of order greater than two in z_1 and z_2.

Proof. We begin by applying Proposition 3.25 to $\exp(-A_1)\exp(A_1 + B_1 + C_1)$, and get that this is equal to

$$\exp\left(B_1 + C_1 + \frac{1}{2}[-A_1, B_1 + C_1] + O_1\right),$$

where O_1 denotes terms of order greater than 2 in z_1 and z_2.

Then we multiply on the right by $\exp(-C_1)$ and get, with Proposition 3.25 again,

$$\exp\left(B_1 + \frac{1}{2}[-A_1, B_1 + C_1] + \frac{1}{2}[B_1, -C_1] + O_2\right),$$

where O_2 denotes terms of order greater than 2 in z_1 and z_2. By definition of D_1, we have proved

$$\exp(-A_1)\exp(z_1 l_1 + z_2 l_2)\exp(-C_1) = \exp\left(B_1 + D_1 + O_2\right).$$

Recall that $D_1 = A_2 + B_2 + C_2$, and that all of these are of degree two in z_1 and z_2. We apply another time the Proposition 3.25, to $\exp(-A_2)\exp(B_1 + D_1 + O_2)$, but here we only need to use the first term in the development of BCH. We might say that A_2 commutes up to order two with elements of degree greater or equal to one in z_1, z_2. We get

$$\exp(-A_2)\exp(B_1 + D_1 + O_2) = \exp(B_1 + B_2 + C_2 + O_3),$$

where O_3 denotes terms of order greater than 2 in z_1 and z_2.

One further use of the Baker-Campbell-Hausdorff formula yields

$$\exp(-A_2)\exp(B_1 + D_1 + O_2)\exp(-C_2) = \exp(B_1 + B_2 + O_4),$$

where O_4 denotes terms of order greater than 2 in z_1 and z_2.

Consider now $\exp(C_2)\exp(C_1)$. Since $C_1, C_2 \in \mathrm{Ad}(\exp(a))(\mathfrak{k})$, we have

$$\exp(C_2)\exp(C_1) = \exp(a)k_2\exp(-a)$$

for some $k_2 \in K$. On the other hand, we have $k_1 := \exp(A_1)\exp(A_2) \in K$.

Summing up we have proved that

$$\exp(z_1 l_1 + z_2 l_2) = k_1 \exp(B_1 + B_2 + O_4)\exp(a)k_2\exp(-a).$$

But then

$$\exp(z_1 l_1 + z_2 l_2)\exp(a) = k_1 \exp(B_1 + B_2 + O_4)\exp(a)k_2,$$

and one last application of Proposition 3.25 gives the lemma, because B_1, B_2 and a commute:

$$\exp(z_1 l_1 + z_2 l_2)\exp(a) = k_1 \exp(B_1 + B_2 + a + O)k_2$$

where O denotes terms of order greater than 2 in z_1 and z_2. □

Lemma 3.28. *We have*

$$H_{l_1,l_2}(a) = \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 f(a + B_2 + B_1).$$

Proof. We first use $K \times K$-invariance of ψ and Lemma 3.27 to write

$$\psi(\exp(z_1 l_1 + z_2 l_2)\exp(a)) = \psi(\exp(a + B_1 + B_2 + O)).$$

Then

$$\begin{aligned} H_{l_1,l_2}(a) &= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_{z_1,z_2=0} \psi(\exp(z_1 l_1 + z_2 l_2)\exp(a)) \\ &= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 \psi(\exp(a + B_1 + B_2 + O)) \end{aligned}$$

because O is of order greater than two, this becomes

$$= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 \psi(\exp(a + B_1 + B_2))$$

since $a + B_1 + B_2 \in \mathfrak{a}$, this is

$$= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 f(a + B_1 + B_2)$$

□

It remains to determine $B_1 + B_2$ for all coefficients of the Hessian, and then to compute the coefficient. For that, since we want to reduce to real coordinates, we recall that if $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$\frac{\partial^2}{\partial z_1 \partial \overline{z_2}} = \frac{1}{4}\left(\frac{\partial^2}{\partial x_1 \partial x_2} + \frac{\partial^2}{\partial y_1 \partial y_2}\right) + \frac{i}{4}\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2}\right).$$

Determining $H_{l_1,l_2}(a)$

Lemma 3.29. *Assume $l_1, l_2 \in \mathfrak{s}$. Then $H_{l_1,l_2}(a)$ is the corresponding coefficient of $\frac{1}{4}\mathrm{Hess}_{\mathbb{R}}(f)(a)$:*

$$H_{l_1,l_2}(a) = \frac{1}{4}\left.\frac{\partial^2}{\partial y_1 \partial y_2}\right|_0 f(a + y_1 i l_1 + y_2 i l_2).$$

Proof. In this case we have $z_1 l_1 + z_2 l_2 = A_1 + B_1 + 0$ with $A_1 = x_1 l_1 + x_2 + l_2 \in \mathfrak{s}$ and $B_1 = y_1 l_1 + y_2 l_2 \in \mathfrak{a}$, and A_1 and B_1 commute, so $D_1 = 0$ and $B_2 = 0$.

Then by Lemma 3.28,

$$\begin{aligned} H_{l_1,l_2}(a) &= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 f(a + y_1 i l_1 + y_2 i l_2) \\ &= \frac{1}{4}\left.\frac{\partial^2}{\partial y_1 \partial y_2}\right|_0 f(a + y_1 i l_1 + y_2 i l_2) \end{aligned}$$

□

Lemma 3.30. *Assume $l_1 \in \mathfrak{k}_\alpha$ and $l_2 \in \mathfrak{s}$, then $H_{l_1,l_2}(a) = 0$.*

Proof. Let us first determine the A_1, B_1, C_1 such that $z_1 l_1 + z_2 l_2 = A_1 + B_1 + C_1$. Using Lemma 3.26, write

$$il_1 = -\coth(\alpha(a))l_1' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1')$$

with $l_1' = \frac{1}{\alpha(a)}\mathrm{ad}(a)(il)$.

Then we have

$$\begin{aligned}
A_1 &= x_1 l_1 - y_1 \coth(\alpha(a))l_1' + x_2 l_2 \\
B_1 &= y_2 i l_2 \\
C_1 &= \frac{y_1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1') = y_1 i l_1 + y_1 \cosh(\alpha(a))l_1'
\end{aligned}$$

We must now compute $D_1 = \frac{1}{2}\left([B_1, A_1] + [C_1, A_1] + [C_1, B_1]\right)$. In fact we must only determine B_2 which is the part of D_1 that lies in $\mathfrak{a}$.

We have

$$\begin{aligned}
[B_1, A_1] &= [y_2 i l_2, x_1 l_1 - y_1 \coth(\alpha(a))l_1' + x_2 l_2] \\
&= -y_1 y_2 \coth(\alpha(a))[il_2, l_1'] + x_1 y_2 [il_2, l_1]
\end{aligned}$$

Now $il_2 \in \mathfrak{a}$ and $l_1, l_1' \in \mathfrak{k}_\alpha \subset \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ so $[il_2, l_1'], [il_2, l_1] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$, and the third point of Lemma 3.26 applies to show that the $\mathfrak{a}$ component of $[B_1, A_1]$ is zero.

For the second part, write

$$\begin{aligned}
[C_1, A_1] =& x_1 y_1 \cosh(\alpha(a))[l_1', l_1] - y_1^2 \coth(\alpha(a))[il_1, l_1'] \\
&+ x_2 y_1 [il_1, l_2] + x_2 y_1 \cosh(\alpha(a))[l_1', l_2]
\end{aligned}$$

We have here $[l_1', l_1], [l_1', l_2] \in \mathfrak{k}$, and $[il_1, l_2] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ as above, so only $[il_1, l_1']$ matters. By the properties of the root decomposition,

$$[il_1, l_1'] \in (\mathfrak{g}_{-2\alpha} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{2\alpha}) \cap i\mathfrak{k}$$

and $\mathfrak{g}_{-2\alpha} = \mathfrak{g}_{2\alpha} = \{0\}$. So $[il_1, l_1'] \in \mathfrak{a}$. But in fact we do not need to determine it more explicitly because it appears as a term in y_1^2 and these are ignored in the computation of $\partial\overline{\partial}$.

For the third part,

$$[C_1, B_1] = y_1 y_2 [il_1, il_2] + y_1 y_2 \cosh(\alpha(a))[l_1', il_2]$$

with $[il_1, il_2] \in \mathfrak{k}$ and $[l_1', il_2] \in \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ so there is no contribution to B_2.

We have thus proved that

$$B_2 = -\frac{1}{2} y_1^2 \coth(\alpha(a))[il_1, l_1'].$$

Lemma 3.28 now gives

$$\begin{aligned}
H_{l_1,l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\bigg|_0 f(a + B_2 + B_1) \\
&= \frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\bigg|_0 f(a + y_2 i l_2 - \frac{1}{2} y_1^2 \coth(\alpha(a))[il_1, l_1']) \\
&= 0
\end{aligned}$$

$\square$

Lemma 3.31. *Assume $l_1 \in \mathfrak{k}_{\alpha_1}$ and $l_2 \in \mathfrak{k}_{\alpha_2}$, with $\alpha_1 \neq \alpha_2$ positive roots. Then $H_{l_1,l_2}(a) = 0$.*

Proof. Using Lemma 3.26, we write

$$\begin{aligned}
il_1 &= -\coth(\alpha_1(a)) l_1' + \frac{1}{\sinh(\alpha_1(a))} (\mathrm{Ad}(\exp(a))(l_1') \\
il_2 &= -\coth(\alpha_2(a)) l_2' + \frac{1}{\sinh(\alpha_2(a))} (\mathrm{Ad}(\exp(a))(l_2')
\end{aligned}$$

with $l_1' = \frac{1}{\alpha_1(a)} \mathrm{ad}(a)(il_1)$ and $l_2' = \frac{1}{\alpha_2(a)} \mathrm{ad}(a)(il_2)$.

Then we have

$$\begin{aligned}
A_1 &= x_1 l_1 + x_2 l_2 - y_1 \coth(\alpha_1(a)) l_1' - y_2 \coth(\alpha_2(a)) l_2' \\
B_1 &= 0 \\
C_1 &= y_1 \frac{1}{\sinh(\alpha_1(a))} (\mathrm{Ad}(\exp(a))(l_1') + y_2 \frac{1}{\sinh(\alpha_2(a))} (\mathrm{Ad}(\exp(a))(l_2')
\end{aligned}$$

and

$$\begin{aligned}
D_1 &= \frac{1}{2}[C_1, A_1] \\
&= \frac{1}{2}[y_1 il_1 + y_2 il_2 + y_1 \cosh(\alpha_1(a)) l_1' + y_2 \cosh(\alpha_2(a)) l_2', A_1]
\end{aligned}$$

We have $y_1 \cosh(\alpha_1(a)) l_1' + y_2 \cosh(\alpha_2(a)) l_2'$ and A_1 in $\mathfrak{k}$, so their bracket remains in $\mathfrak{k}$ and does not appear in B_2. We compute $[y_1 il_1 + y_2 il_2, A_1]$ which is equal to

$$\begin{aligned}
&x_2 y_1 [il_1, l_2] - y_1^2 \coth(\alpha_1(a))[il_1, l_1'] - y_1 y_2 \coth(\alpha_1(a))[il_2, l_1'] \\
&+ x_1 y_2 [il_2, l_1] - y_2^2 \coth(\alpha_2(a))[il_2, l_2'] - y_1 y_2 \coth(\alpha_2(a))[il_1, l_2'].
\end{aligned}$$

Again the properties of the root decomposition tell us that $[il_1, l_2]$, $[il_1, l_2']$, $[il_2, l_1]$, and $[il_2, l_1']$ are in $\bigoplus_{\alpha \in \Phi} \mathfrak{g}_\alpha$, so the corresponding terms do not contribute to B_2. As before, $[il_1, l_1']$ and $[il_2, l_2']$ are in $\mathfrak{a}$, so we get

$$B_2 = \frac{1}{2}(-y_1^2 \coth(\alpha_1(a))[il_1, l_1'] - y_2^2 \coth(\alpha_2(a))[il_2, l_2']).$$

Applying Lemma 3.28, we get

$$\begin{aligned}
H_{l_1,l_2}(a) &= \frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\bigg|_0 f(a + B_2 + B_1) \\
&= \frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\bigg|_0 f(a - \frac{1}{2}(y_1^2 \coth(\alpha_1(a))[il_1, l_1'] + y_2^2 \coth(\alpha_2(a))[il_2, l_2'])) \\
&= 0
\end{aligned}$$

□

Suppose now that $\alpha_1 = \alpha_2 = \alpha$. The subspace $\mathfrak{k}_\alpha$ is two dimensional, and we have chosen a basis formed by the vectors $e_\alpha + \theta(e_\alpha)$ and $ie_\alpha - i\theta(e_\alpha)$.

First we deal with the case when $l_1 \neq l_2$.

Lemma 3.32. *Suppose $l_1 = e_\alpha + \theta(e_\alpha)$ and $l_2 = ie_\alpha - i\theta(e_\alpha)$. Then*

$$H_{l_1,l_2}(a) = \frac{i}{2}\alpha(\nabla f(a)),$$

and

$$H_{l_2,l_1}(a) = -\frac{i}{2}\alpha(\nabla f(a)).$$

Proof. Using Lemma 3.26, we write, just as in the previous proof

$$\begin{aligned}
il_1 &= -\coth(\alpha(a))l_1' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1') \\
il_2 &= -\coth(\alpha(a))l_2' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_2')
\end{aligned}$$

with

$$\begin{aligned}
l_1' &= l_2 \\
l_2' &= -l_1
\end{aligned}$$

Then we have

$$\begin{aligned}
A_1 &= (x_1 + y_2 \coth(\alpha(a)))l_1 + (x_2 - y_1 \coth(\alpha(a)))l_2 \\
B_1 &= 0 \\
C_1 &= y_1 \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1') + y_2 \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_2')
\end{aligned}$$

and

$$\begin{aligned}
D_1 &= \frac{1}{2}[C_1, A_1] \\
&= \frac{1}{2}[y_1 il_1 + y_2 il_2 + y_1 \cosh(\alpha(a))l_2 - y_2 \cosh(\alpha(a))l_1, A_1]
\end{aligned}$$

Once again the bracket of $y_1\cosh(\alpha(a))l_2 - y_2\cosh(\alpha(a))l_1$ with A_1 yields only terms in $\mathfrak{k}$ so we compute $[y_1 il_1 + y_2 il_2, A_1]$, which is equal to

$$y_2(x_1 + y_2\coth(\alpha(a)))[il_2, l_1] + y_1(x_2 - y_1\coth(\alpha(a)))[il_1, l_2].$$

Using the explicit choices of l_1 and l_2 we have

$$\begin{aligned} -[il_2, l_1] = [il_1, l_2] &= [i(e_\alpha + \theta(e_\alpha)), ie_\alpha - i\theta(e_\alpha)] \\ &= [ie_\alpha, -i\theta(e_\alpha)] + [i\theta(e_\alpha), ie_\alpha] \\ &= 2[e_\alpha, \theta(e_\alpha)] \\ &= -2[e_\alpha, e_{-\alpha}] \\ &= -2h_\alpha. \end{aligned}$$

Finally we have

$$B_2 = (y_2x_1 + y_2^2\coth(\alpha(a)) - y_1x_2 + y_1^2\coth(\alpha(a)))h_\alpha.$$

Applying Lemma 3.28, we get

$$\begin{aligned} H_{l_1,l_2}(a) &= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 f(a + B_2 + B_1) \\ &= \frac{i}{4}\left.\left(\frac{\partial^2}{\partial x_1 \partial y_2} - \frac{\partial^2}{\partial y_1 \partial x_2}\right)\right|_0 f(a + (y_2x_1 - y_1x_2)h_\alpha) \\ &= \frac{i}{2}(Df)_a(h_\alpha) \end{aligned}$$

where $(Df)_a$ denotes the differential of f at a, so

$$H_{l_1,l_2}(a) = \frac{i}{2}\langle h_\alpha, \nabla f(a)\rangle$$

by definition of h_α, this is

$$H_{l_1,l_2}(a) = \frac{i}{2}\alpha(\nabla f(a)).$$

$\square$

Lemma 3.33. *Suppose now that $l_1 = l_2 = e_\alpha + \theta(e_\alpha)$, then*

$$H_{l_1,l_2}(a) = \frac{1}{2}\alpha(\nabla f(a))\coth(\alpha(a)).$$

Proof. Using Lemma 3.26, we write

$$il_2 = il_1 = -\coth(\alpha(a))l_1' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1')$$

with

$$l_2' = l_1' = ie_\alpha - i\theta(e_\alpha)$$

Then we have

$$\begin{aligned}
A_1 &= (x_1 + x_2)l_1 - (y_1\coth(\alpha(a)) + y_2\coth(\alpha(a)))l_1' \\
B_1 &= 0 \\
C_1 &= y_1\frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1') + y_2\frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_2')
\end{aligned}$$

and

$$\begin{aligned}
D_1 &= \frac{1}{2}[C_1, A_1] \\
&= \frac{1}{2}[(y_1 + y_2)il_1 + (y_1\cosh(\alpha(a)) + y_2\cosh(\alpha(a)))l_1', A_1]
\end{aligned}$$

Once again the bracket of $(y_1\cosh(\alpha(a)) + y_2\cosh(\alpha(a)))l_1'$ with A_1 yields only terms in $\mathfrak{k}$, so we just compute

$$[(y_1 + y_2)il_1, A_1] = -(y_1 + y_2)^2\coth(\alpha(a))[il_1, l_1'].$$

Using the explicit choices of l_1 we have

$$\begin{aligned}
[il_1, l_1'] &= [i(e_\alpha + \theta(e_\alpha)), ie_\alpha - i\theta(e_\alpha)] \\
&= -2h_\alpha.
\end{aligned}$$

Finally we have

$$B_2 = (y_1 + y_2)^2\coth(\alpha(a))h_\alpha.$$

Applying Lemma 3.28, we get

$$\begin{aligned}
H_{l_1,l_2}(a) &= \left.\frac{\partial^2}{\partial z_1 \partial \overline{z_2}}\right|_0 f(a + B_2 + B_1) \\
&= \frac{1}{4}\left.\frac{\partial^2}{\partial y_1 \partial y_2}\right|_0 f(a + (y_1^2 + 2y_1y_2 + y_2^2)\coth(\alpha(a))h_\alpha) \\
&= \frac{\coth(\alpha(a))}{2}(Df)_a(h_\alpha) \\
&= \frac{\coth(\alpha(a))}{2}\langle h_\alpha, \nabla f(a)\rangle \\
&= \frac{\coth(\alpha(a))}{2}\alpha(\nabla f(a)).
\end{aligned}$$

□

The last step is to compute the coefficient of the Hessian with $l_1 = l_2 = ie_\alpha - i\theta(e_\alpha)$, and the result is exactly the same as in the previous case:

Lemma 3.34. *Assume that $l_1 = l_2 = ie_\alpha - i\theta(e_\alpha)$, then*

$$H_{l_1,l_2}(a) = \frac{1}{2}\alpha(\nabla f(a))\coth(\alpha(a)).$$

Proof. Using Lemma 3.26, we write

$$il_2 = il_1 = -\coth(\alpha(a))l_1' + \frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1')$$

with

$$l_2' = l_1' = -e_\alpha - \theta(e_\alpha)$$

The beginning of the computation does not change: we have

$$\begin{aligned}
A_1 &= (x_1 + x_2)l_1 - (y_1\coth(\alpha(a)) + y_2\coth(\alpha(a)))l_1' \\
B_1 &= 0 \\
C_1 &= y_1\frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_1') + y_2\frac{1}{\sinh(\alpha(a))}(\mathrm{Ad}(\exp(a))(l_2') \\
D_1 &= \frac{1}{2}[(y_1 + y_2)il_1 + (y_1\cosh(\alpha(a)) + y_2\cosh(\alpha(a)))l_1', A_1]
\end{aligned}$$

Once again the bracket of $(y_1\cosh(\alpha(a)) + y_2\cosh(\alpha(a)))l_1'$ with A_1 yields only terms in $\mathfrak{k}$, so we just compute

$$[(y_1 + y_2)il_1, A_1] = -(y_1 + y_2)^2\coth(\alpha(a))[il_1, l_1'].$$

Now the expression of l_1 has changed, but we have

$$\begin{aligned}
[il_1, l_1'] &= [i(ie_\alpha - i\theta(e_\alpha)), -e_\alpha - \theta(e_\alpha)] \\
&= 2[e_\alpha, \theta(e_\alpha)] \\
&= -2h_\alpha.
\end{aligned}$$

In other words we have again

$$B_2 = (y_1 + y_2)^2\coth(\alpha(a))h_\alpha,$$

and applying Lemma 3.28, we get again

$$\begin{aligned}
H_{l_1,l_2}(a) &= \left.\frac{\partial^2}{\partial z_1\partial\overline{z_2}}\right|_0 f(a + B_2 + B_1) \\
&= \frac{\coth(\alpha(a))}{2}\alpha(\nabla f(a)).
\end{aligned}$$

□

Chapter 4

Group compactifications and metrics

In this chapter we define the group compactifications, and derive from the theory classifying such varieties the information that we will use to get our results. To each group compactification, equipped with a polarization, *i.e.* an ample linearized line bundle on it, is associated a polytope that contains all the information about the compactification. In particular, this polytope determines the asymptotic behavior of the potentials of metrics on the big orbit.

We begin by a short section describing line bundles on a reductive group G that are linearized by $G \times G$. This allows to introduce the notion of linearized line bundle and will be used to define the potentials of metrics on the big orbit G of a compactification of G.

Then we give a brief overview of the theory of toric varieties, which is used then to study group compactification. Indeed, a group compactification admits a toric subvariety that contains all the information about the compactification. If we have a polarization, it restricts to a polarization on the toric subvariety, and thus translates as the data of an integral polytope by the classical theory of projective toric varieties.

We also provide a description of examples of group compactification, the most important class of such being the wonderful compactifications. They turn out to be Fano manifolds and we describe the polytope associated to their anticanonical polarization.

We then introduce the different notions of potential of a hermitian metric on a line bundle, and use the polytopes to describe the asymptotic behavior of potentials of metrics on the group orbit. Combining the asymptotic description, the KAK integration formula and the computation of the complex Monge-Ampère from the first chapter, we see how we can recover, up to a constant, the formula for the degree of an ample line bundle on a group compactification which was computed by Kazarnovskii [Kaz87] and Brion [Bri89].

4.1 Linearized line bundles on reductive groups

First we define the general notion of G-linearized bundle on a G-variety, for G any Lie group.

Definition 4.1. A G-*linearization* of a line bundle L on a G-variety X is a G-action on L such that:
- the G-action on L lifts the G-action on X, and
- the map between the fibers L_x and $L_{g\cdot x}$ defined by the action of $g \in G$ is linear.

Let now G be a connected reductive group again. The group $G \times G$ acts on G through the actions of G by left and right translations on itself: $(g_1, g_2) \cdot g = g_1 g g_2^{-1}$.

Proposition 4.2. *The $G \times G$-linearized line bundles on G are classified by characters of G. Furthermore, any $G \times G$-linearized line bundle on G admits a $G \times \{e\}$ equivariant trivialization.*

In particular, if G is semisimple, any $G \times G$-linearized line bundle on G admits a $G \times G$-equivariant trivialization.

Proof. Let L be a $G \times G$-linearized line bundle on G. We first prove that it admits a left-G-equivariant trivialization.

Choose a nonzero element 1_e in the fiber L_e over the neutral element $e \in G$. The section s defined by

$$s(g) = (g, e) \cdot 1_e$$

is well defined on all G, because the left action is simply transitive, and non zero everywhere because each (g, e) induces a linear isomorphism from L_e to L_g, so s trivializes L. It is also clearly $G \times \{e\}$ equivariant by construction.

Associate to L the character of G defined as the character of the linear action of $\mathrm{diag}(G) \subset G \times G$ on L_e.

Conversely, given a character χ of G, we get a $G \times G$-linearized bundle on G by considering the trivial bundle $G \times \mathbb{C}$ together with the action of $G \times G$ defined by

$$(g_1, g_2) \cdot (g, t) = (g_1 g g_2^{-1}, -\chi(g_2)t).$$

The construction of s above gives $s(g) = (g, 1)$, and we recover the character χ as the character of the action of $\mathrm{diag}(G)$ on L_e. $\square$

More generally one can get a similar result for homogeneous spaces, see [KKLV89]. We can also determine if all line bundles on G can be $G \times G$-linearized. This is the case if G is simply connected, and in general if G_{sc} is the simply connected group above G, then every line bundle on G is $G_{sc} \times G_{sc}$-linearized. This is also explained in [KKV89, KKLV89].

4.2 Polarized toric varieties

4.2.1 Toric varieties and lattice polytopes

In this section we provide some general results about toric varieties, that either have found an analogue in the case of group compactifications, or will be used in their study. General references for toric varieties include [CLS11, Ful93, Oda88].

Let $T \simeq (\mathbb{C}^*)^r$ be a torus, denote by M, respectively N, its group of characters, respectively algebraic one parameter subgroups.

Definition 4.3. A *polarized toric variety* (X, L) of dimension r is a normal projective T-variety X with an open dense orbit isomorphic to T, equipped with a T-linearized ample line bundle L.

Let us recall the theorem that classifies such objects combinatorially. The precise correspondence will be progressively explained. In the next section we will see how Alexeev and Katzarkov [AK05], building on the work of Alexeev and Brion [AB04a, AB04b], extended this theorem to group compactifications.

Theorem 4.4. *Polarized toric variety (X, L) are in bijective correspondence with convex, full-dimensional lattice polytopes P in $M \otimes \mathbb{R}$.*

By lattice polytope we mean that the vertices of P are in M.

In fact, the underlying toric variety X is fully determined by the *normal fan* Σ of the polytope P, defined as follows.

Given a vertex $v \in P$, consider the (closed, full-dimensional) cone $C_v \subset M \otimes \mathbb{R}$ with vertex 0, generated by the lattice points of $-v + P$, the translate of P by $-v$. Let σ_v denote the dual cone to C_v, i.e.

$$\sigma_v = \{n \in N \otimes \mathbb{R}; m(n) \geq 0 \ \forall m \in C_v\}.$$

The normal fan Σ of P is the collection of cones which consists of the cones σ_v and their faces. Of course some of these cones have common faces that must be counted only once. This collection of cones satisfies the following two conditions [CLS11, Theorem 2.3.2]:

– for any $\sigma \in \Sigma$, the faces of σ are in Σ, and

– the intersection of σ with another cone in Σ is a union of faces of σ.

In general a collection of convex rational polyhedral cones satisfying these two conditions is called a *fan* and corresponds to a toric variety, not necessarily projective.

One of the major relationship between a toric variety and its fan is the orbit-cone correspondence, which we recall here along with the polytope version. By convention, a polytope with non-empty interior in $M \otimes \mathbb{R}$ has a unique face of dimension r, which is itself.

Proposition 4.5. *There is a bijective correspondence between the following:*

– *the T-orbits in X of complex dimension k;*

– *the cones in Σ of real* codimension *k in $N \otimes \mathbb{R}$;*

– the faces of P of real dimension k in $M \otimes \mathbb{R}$.
Furthermore, in each case we can define a partial order by saying that an orbit (resp. cone, face) is smaller than another one if it is in its closure. Then the correspondence between orbits and faces is order-preserving, and it reverses the order between orbits and cones.

The polytope adds the information about the ample line bundle to the fan. More generally we describe the combinatorial data associated to a linearized line bundle on a toric variety (assumed to be projective here).

4.2.2 Line bundles

Let X be a toric variety as above. Let L be a T-linearized line bundle on X.

For any T-fixed point x on X, T acts linearly on the fiber of L at x. Denote by v_x the opposite of the character of this action. We define the *support function* g_L of the line bundle L as the piecewise linear function on $N \otimes \mathbb{R}$, which takes the value $v_x(n)$ at a point n in the closure of the cone of dimension r corresponding to the fixed point x by the orbit-cone correspondence.

We can also associate to L the polytope P_L defined by

$$P_L = \{m \in M \otimes \mathbb{R}; g_L(n) \leq m(n) \ \forall n \in N \otimes \mathbb{R}\}.$$

The lattice points of P_L determine a basis of the space $H^0(X, L)$ of algebraic sections of L [CLS11, Proposition 4.3.3]. More precisely, if s_0 denotes a T-equivariant section of L, then the sections s_m defined by $s_m(t) = m(t)s_0(t)$ on T for $m \in P_L \cap M$ extend to X and form a basis of algebraic sections of L.

One can characterize the ampleness of L in terms of its support function g_L.

Proposition 4.6. *[CLS11, Lemma 6.1.13] The line bundle L is ample if and only if g_L is concave, and $v_x \neq v_y$ for any two different fixed points x and y in X.*

There are several properties of the line bundles (and more generally divisors) on the toric variety that can be read off from the associated polytope or support function, see [CLS11].

The polytope associated to a polarized toric variety (X, L) is P_L. This explains one direction of the correspondence.

We attract the reader's attention to the fact that the support function of the line bundle is *not* the support function of the polytope P_L. In fact the support function of the polytope P_L will be more important to us. It is the convex function $v_L : N \otimes \mathbb{R} \to \mathbb{R}$ defined by

$$v(x) = \sup\{m(x); m \in P_L\}.$$

In the case when L is nef, we have $v(x) = -g_L(-x)$ so the data of v_L is equivalent to the data of g_L or L. Furthermore, this function is piecewise linear with respect to the opposite of the fan Σ.

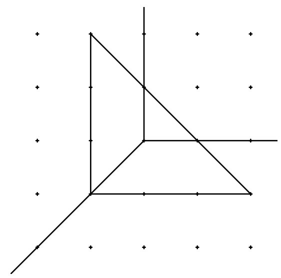


Figure 4.1: Polytope and fan of $\mathbb{P}^2$

Example 4.7. Consider the complex projective line $\mathbb{P}^1$. It is a toric Fano manifold. The polytope associated to the anticanonical line bundle is $[-1,1]$ which has vertices in the lattice $\mathbb{Z}$. The support function of $-K_{\mathbb{P}^1}$ is $x \mapsto -|x|$ and the support function of the polytope $[-1,1]$ is $x \mapsto |x|$.

Example 4.8. Figure 4.1 gives the fan of $\mathbb{P}^2$ and the polytope corresponding to the anticanonical line bundle. The support function of the anticanonical line bundle is linear on each cone of the fan, equal to $-x-y$ when $x, y \geq 0$, to $2x-y$ when $x \leq 0$, $y \geq x$, and to $2y-x$ when $y \leq 0$, $x \geq y$. The support function of the polytope is linear on the opposite of these cones: it is $-x-y$ when $x, y \leq 0$, $2x-y$ when $x \geq 0$, $y \leq x$, and $2y-x$ when $y \geq 0$, $x \leq y$.

4.2.3 Smoothness criterion

The smoothness of a toric variety is an information that can be easily seen on the fan, or on the associated polytope for a polarized variety. Let us first recall the definition of a Delzant polytope before stating the criterion.

Definition 4.9. A full-dimensional convex lattice polytope $P \subset M \otimes \mathbb{R}$ is called *Delzant* if the slopes of the edges at each vertex form a basis of M.

We will also call a cone *smooth* if it is generated by a part of a basis of N.

It is clear that a polytope is Delzant if and only if all the full dimensional cones of its normal fan are smooth, and this implies that all cones of the normal fan are smooth.

Proposition 4.10. *[CLS11, Theorem 3.1.19] Given a polarized toric variety (X, L) with associated polytope P, the following are equivalent:*
- *X is smooth ;*
- *the polytope P is Delzant ;*
- *all the cones of the normal fan of P are smooth.*

4.3 Group compactifications

4.3.1 Definition

Let G be a connected complex reductive group.

Definition 4.11. A normal irreducible projective $G \times G$-variety X is called a *$G \times G$-equivariant compactification* of G if X admits an open and dense orbit under $G \times G$, equivariantly isomorphic to G on which $G \times G$ acts by left and right multiplication.

If X is a $G \times G$-equivariant compactification of G, we will always identify G with the open and dense orbit in X. We will more succinctly call X a *group compactification* of G.

Remark 4.12. Recall that a *spherical variety* under the group G is a G-variety on which a Borel subgroup B of G acts with an open orbit. Here we are considering $G \times G$-varieties, so we consider the $B \times B$-orbits. It suffices to look at the $B \times B$-orbits in G. These are called the Bruhat cells and the Bruhat decomposition shows that there is an open Bruhat cell, namely BB^- where B^- is the opposite Borel subgroup. So group compactifications are spherical varieties.

Now choose T a maximal torus in G.

Proposition 4.13. *[BK05, Corollary 6.2.14] Let X be an equivariant group embedding of G, then the closure Z of T in X is a normal toric variety.*

We use as reference the sixth chapter in [BK05], which is a convenient reference for the results on group compactifications we will use. The references to the original papers can be found in this book.

4.3.2 Group compactifications and polytopes

We can now give the generalization of the correspondence between polarized varieties and polytopes to the setting of group compactifications.

Let G be a reductive group, and X a compactification of G. Choose T a maximal torus in G and denote by Z the closure of T in X. We have seen that Z is a toric variety. This toric subvariety admits in addition an action of the Weyl group W of G. Let L be an ample $G \times G$-linearized line bundle on X. The restriction of L to Z is a line bundle, linearized by the normalizer $N_{G\times G}(T)$.

Theorem 4.14. *[[AK05, Theorem 2.4], based on [AB04a, AB04b]] The restriction of L to Z is an ample line bundle, and this gives a W-invariant lattice polytope P associated to the polarized group compactification (X, L). Conversely, given a W-invariant full-dimensional lattice polytope P, there exists a polarized $G \times G$-equivariant compactification of G whose associated polytope is P.*

This comes with a kind of orbit-face correspondence again.

Proposition 4.15. *Let (X, L) be a polarized compactification of G, with associated polytope P. The $G \times G$ orbits in X are in an order preserving bijective correspondence with the W-orbits of faces of the polytope P.*

Let us now explain one way to recover a group compactification from its polytope, remark that this works in particular for toric varieties, for which we did not explain this direction yet either. This general construction is again taken from [AB04a, AB04b, AK05].

Recall first that the algebra of regular functions on G is described as a $G \times G$-representation in [Tim11, Theorem 2.15] by:

$$\mathbb{C}[G] \simeq \bigoplus_{\lambda \in M \cap (\mathfrak{a}^*)^+} \mathrm{End}(E_\lambda)$$

where E_λ is the finite dimensional irreducible representation of G of highest weight the dominant weight λ, and $\mathrm{End}(E_\lambda) = E_\lambda^* \otimes E_\lambda$ is the space of endomorphisms of E_λ, and is an irreducible $G \times G$-representation. More precisely, let us describe how this isomorphism is realized. An element $\sigma \otimes x \in E_\lambda^* \otimes E_\lambda$ defines a matrix coefficient $f_{\sigma \otimes x}$ of the representation E_λ:

$$f_{\sigma \otimes x}(g) = \sigma(g \cdot x)$$

which is an element of $\mathbb{C}[G]$.

Given a W-invariant full dimensional lattice polytope P in $M \otimes \mathbb{R}$, define P^+ to be the part of P lying in the positive Weyl chamber of $\mathfrak{a}^*$. Let $\mathcal{C}$ be the cone over $(1, P^+)$ in $\mathbb{Z} \oplus M_\mathbb{R}$. The vector space

$$R_P := \bigoplus_{\mu \in \mathcal{C} \cap (\mathbb{Z} \oplus M)} \mathrm{End}(F_\mu) \subset \mathbb{C}[\mathbb{C}^* \times G]$$

has a natural structure of subalgebra, and is finitely generated. We can thus define $X := \mathrm{Proj}(R_P)$ and a coherent sheaf $L = \mathcal{O}(1)$. This X is in fact a compactification of G and L is a $G \times G$-linearized ample line bundle on X whose associated polytope is P.

The polytope P of (X, L) also encodes the structure of the space of sections of L as a $G \times G$ representation. Namely, we have:

$$H^0(X, L) = \bigoplus_{\lambda \in P^+ \cap M} \mathrm{End}(V_\lambda).$$

4.3.3 Smoothness criterion

Let us now give the partial smoothness criterion obtained by Alexeev and Katzarkov, where we again restrict to the case of group compactifications.

Proposition 4.16. *[AK05, Proposition 2.5]*

- *If X is smooth then the associated polytope is Delzant.*
- *If P is Delzant, and no vertex of P lies in a Weyl wall, then X is smooth.*

Remark 4.17. In the second case, the added condition that no vertex of P lies in a Weyl wall ensures that X is toroidal as a spherical variety, *i.e.* that no $B\times B$-stable, not $G \times G$-stable divisor of X contains a closed $G \times G$-orbit. Toroidal, smooth compactifications of groups are also called regular compactifications of groups. For toroidal compactifications of groups, it was already known that the smoothness of the group compactification and of its toric subvariety were equivalent.

We turn now to some examples of group compactifications. We already reviewed the toric varieties, which are compactifications of groups with no semisimple part. On the opposite end, the most known family of such varieties consists of the wonderful compactifications of semisimple adjoint groups.

We will present these in the following section, including a description of the line bundles on them. We do not describe in general all line bundles on group compactifications to avoid lengthening the text too much, but the Picard group of any spherical variety was described by Michel Brion in [Bri89], and a description of the line bundles on regular compactifications even more similar to the one given for toric varieties is possible, see [Bif90].

4.4 Wonderful compactifications

4.4.1 Definition and existence

A wonderful compactification can be defined by some of its remarkable properties.

Definition 4.18. A $G \times G$-equivariant compactification X of G is called wonderful if it satisfies the conditions:

- X is smooth;
- $X \setminus G$ is the union of smooth normal crossing prime divisors, with nonempty intersections;
- the $G \times G$-orbits in X are precisely the intersections of families of these divisors.

The existence of such a compactification for a semisimple adjoint group was proved by de Concini and Procesi in [DCP83]. In fact, they considered compactifications of symmetric spaces under a semisimple adjoint group, but we focus here on group compactifications.

Theorem 4.19. *[DCP83] If G is a semisimple adjoint group then G admits a unique wonderful compactification.*

There exists several constructions of wonderful compactifications. Given any regular and dominant weight λ in M, if P denotes the convex hull of the images of λ by W, then the compactification of G associated to P is the wonderful compactification of G. We will see that this gives all polarizations, and which one corresponds to the anticanonical line bundle.

There is in fact a simpler way to describe the wonderful compactification given a regular and dominant weight λ. Let E_λ be the irreducible representation of G with highest weight λ. Consider the $G \times G$ projective representation $\mathbb{P}(\mathrm{End}(E_\lambda))$. Then the closure in $\mathbb{P}(\mathrm{End}(E_\lambda))$ of the orbit of the identity in $\mathrm{End}(E_\lambda)$ is the wonderful compactification of G.

4.4.2 Line bundles

Let $G = G_{ad}$ be a semisimple adjoint group and X the wonderful compactification of G. Let M_{sc} be the lattice generated by the fundamental weights of the root system Φ of $\mathfrak{g}$. A priori $M \subset M_{sc}$ is different from M_{sc}. Indeed, for an adjoint group G, the lattice of weights (or characters of $T \subset G$) is generated by the roots of Φ.

Let G_{sc} be the corresponding simply connected group. The variety X is also spherical under the $G_{sc} \times G_{sc}$-action induced by the $G_{ad} \times G_{ad}$-action. The only closed orbit is isomorphic to the full flag variety

$$(G_{sc} \times G_{sc})/(B_{sc} \times B_{sc}) = G_{sc}/B_{sc} \times G_{sc}/B_{sc}$$

for $G_{sc} \times G_{sc}$. The restriction map from the Picard group of X to the Picard group of the closed orbit is injective.

Recall that to a character of B_{sc} is associated a G_{sc}-linearized line bundle on G_{sc}/B_{sc}, in the following way: if V_λ denotes the one-dimensional representation of B_{sc} associated to the character λ, one gets a line bundle on G_{sc}/B_{sc} by considering the fiber product $G_{sc} \times_{B_{sc}} V_{-\lambda}$. Any $G_{sc} \times G_{sc}$-linearized line bundle on $G_{sc} \times G_{sc}/B_{sc} \times B_{sc}$ is of the form $\mathcal{L}(\lambda, \mu)$ where λ and μ are two characters of B_{sc} and $\mathcal{L}(\lambda, \mu)$ is the tensor product of the pullbacks by the two projections to G_{sc}/B_{sc} of the corresponding line bundles on G_{sc}/B_{sc}.

The image of the Picard group of X consists of the line bundles of the form $\mathcal{L}(-w_0\lambda, \lambda)$ where w_0 is the element of the Weyl group of G sending the positive Weyl chamber to the negative one. Furthermore, properties of the line bundles are encoded in the corresponding character λ. This is summarized in the following proposition.

Theorem 4.20. *[BK05, Proposition 6.1.11] The Picard group of X is isomorphic to the group M_{sc}. If $\lambda \in M_{sc}$, denote by $L(\lambda)$ the associated line bundle. Then we have also:*
- *$L(\lambda)$ is globally generated if and only if λ is a dominant weight ;*
- *$L(\lambda)$ is ample if and only if λ is dominant and regular ;*
- *$L(\lambda)$ can be $G \times G$ linearized if and only if $\lambda \in M$.*

In fact, all line bundles on X can be $G_{sc} \times G_{sc}$-linearized, where G_{sc} is the simply connected semisimple group over G.

These manifolds are especially interesting for our purposes because they are Fano. The following result gives the polytope associated with the anticanonically polarized wonderful compactification. Recall that given a root system Φ with positive roots Φ^+, ρ denotes the sum of the fundamental weights of Φ, and is also equal to half the sum of the positive roots. In particular, $2\rho \in M$.

Proposition 4.21. *[BK05, Proposition 6.1.11] Let G be a semisimple adjoint group, and X its wonderful compactification. Then the fan corresponding to the toric subvariety $Z \subset X$ is given by the subdivision of $\mathfrak{a}$ induced by the Weyl chambers and their faces. The anticanonical line bundle of X is associated with the weight $2\rho + \sum_{i=1}^{r} \alpha_i$ where α_i for $1 \leq i \leq r$ are the simple roots of Φ^+. In other words, $-K_X = L(2\rho + \sum_{i=1}^{r} \alpha_i)$. This implies that X is Fano, and that the polytope associated to the anticanonically polarized $G \times G$-equivariant compactification $(X, -K_X)$ is the convex hull of the images by W of $2\rho + \sum_{i=1}^{r} \alpha_i$.*

Remark 4.22. The anticanonical line bundle on G/B is associated to the character 2ρ. On the other hand, the character of the action of T on the fixed point of the affine toric variety define by the Weyl chamber is the sum of the simple roots (which are the generators of the dual cone of the Weyl chamber). The proposition shows that the character corresponding to the anticanonical line bundle on X is the sum of these two.

4.4.3 Rank one example: $\mathbb{P}^3$

The only adjoint semisimple group of rank one is $\mathrm{PGL}_2(\mathbb{C})$. Its wonderful compactification is the projective space $\mathbb{P}^3$. Let us describe in details this example.

Consider $\mathbb{P}^3$ as $\mathbb{P}(\mathcal{M}_2(\mathbb{C}))$ by identifiying a two times two matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with the point of $\mathbb{C}^4$ with coordinates (a, b, c, d). Then $\mathrm{PGL}_2(\mathbb{C})$ is the open set

$$\{[a : b : c : d]; ad - bc \neq 0\} \subset \mathbb{P}^3.$$

Furthermore, $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ acts on $\mathbb{P}(\mathcal{M}_2(\mathbb{C}))$ with two orbits determined by the rank of the representatives. The first orbit is precisely the open set $\mathrm{PGL}_2(\mathbb{C})$ and the second the closed set formed by the classes of rank one matrices:

$$\{[a : b : c : d]; ad - bc = 0\}.$$

This closed orbit is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^3$, and it turns out that this identification is also equivariant under $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ once we remark that $\mathbb{P}^1$ is the flag manifold of $\mathrm{PGL}_2(\mathbb{C})$.

It is then clear that $\mathbb{P}^3$ is the wonderful compactification of $\mathrm{PGL}_2(\mathbb{C})$.

The group $\mathrm{PGL}_2(\mathbb{C})$ is of rank one, and we can choose as maximal torus the set $T := \{[a : 0 : 0 : d]; ad \neq 0\}$ formed by the classes of diagonal matrices.

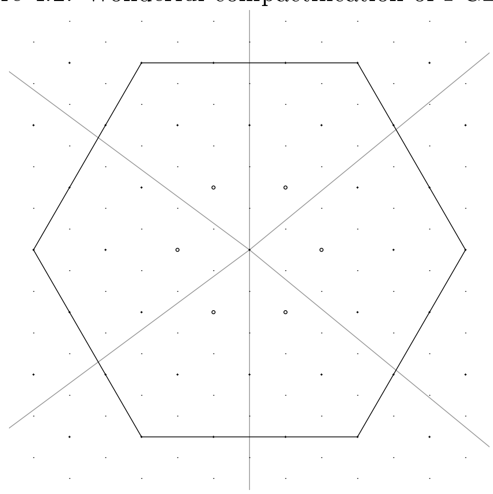
The closure of T in $\mathbb{P}^3$ is $\{[a : b : c : d]; b = c = 0\}$, and is isomorphic to $\mathbb{P}^1$.

4.4.4 Rank two examples

For each root sytem of rank two, there is a corresponding adjoint semisimple group of rank two. For $A_1 \times A_1$, the adjoint group is $\mathrm{PGL}_2(\mathbb{C}) \times \mathrm{PGL}_2(\mathbb{C})$ and the corresponding wonderful compactification is the product $\mathbb{P}^3 \times \mathbb{P}^3$, or by the previous example the product of two copies of the wonderful compactification of $\mathrm{PGL}_2(\mathbb{C})$.

Consider now the root system A_2. The corresponding adjoint semisimple group is $\mathrm{PGL}_3(\mathbb{C})$. The polytope corresponding to its wonderful compactification is given in Figure 4.2. The toric subvariety in this case is the blow up of $\mathbb{P}^2$ at the three torus-fixed points.

Figure 4.2: Wonderful compactification of $\mathrm{PGL}_3(\mathbb{C})$



For type B_2, the adjoint group is $\mathrm{SO}_5(\mathbb{C})$ and the corresponding polytope is represented in Figure 4.3.

Finally the polytope of the wonderful compactification of G_2 is represented in Figure 4.4.

4.4.5 Wonderful compactifications of non adjoint semisimple groups

In rank one, the simply connected group $\mathrm{SL}_2(\mathbb{C})$ admits a wonderful compactification. This is the quadric in $\mathbb{P}^4 = \mathbb{P}(M_2(\mathbb{C}) \oplus \mathbb{C})$ defined by the equation $\det(A) = t^2$ for $(A : t) \in \mathbb{P}(M_2(\mathbb{C}) \oplus \mathbb{C})$.

In higher ranks, Gandini and Ruzzi [GR13] proved that the only simple non adjoint group which admits a wonderful compactification is the symplectic group $\mathrm{Sp}_{2r}(\mathbb{C})$. For example, the polytope corresponding to the wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$ is represented in Figure 4.5.

4.4.6 Automorphism group

Michel Brion determined the automorphism group of the wonderful compactification of a semisimple adjoint group in [Bri07, Example 2.4.5]. The most

Figure 4.3: Wonderful compactification of $\mathrm{SO}_5(\mathbb{C})$

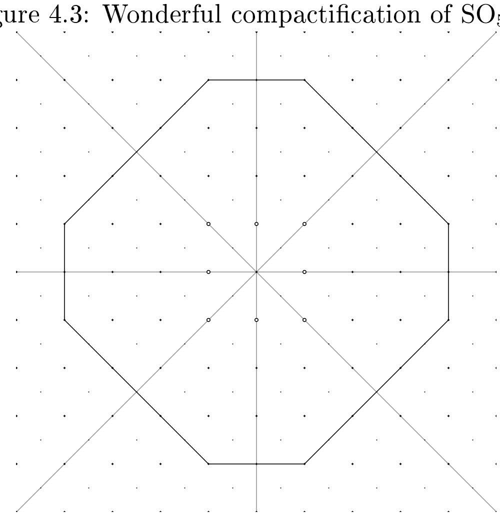


Figure 4.4: Wonderful compactification of G_2

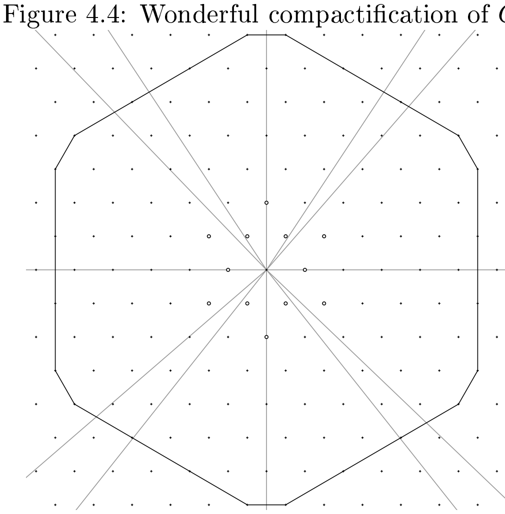
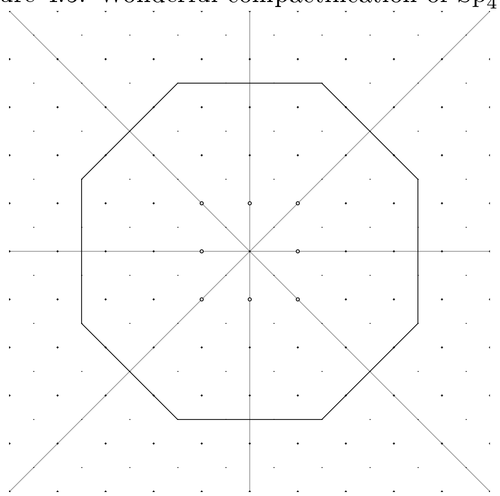


Figure 4.5: Wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$



relevant part for us is the connected component of the identity. Let us recall the result.

Let G_{ad} be a semisimple adjoint group and G the corresponding simply connected group. Let X be the wonderful compactification of G_{ad}. Write $G = (\mathrm{SL}_2(\mathbb{C}))^n \times G'$ where G' contains no direct factor isomorphic to SL_2. Then $\mathrm{Aut}^0(X) \simeq (\mathrm{PSL}_4)^n \times (G'_{ad} \times G'_{ad})$ where G'_{ad} is the adjoint group corresponding to G'.

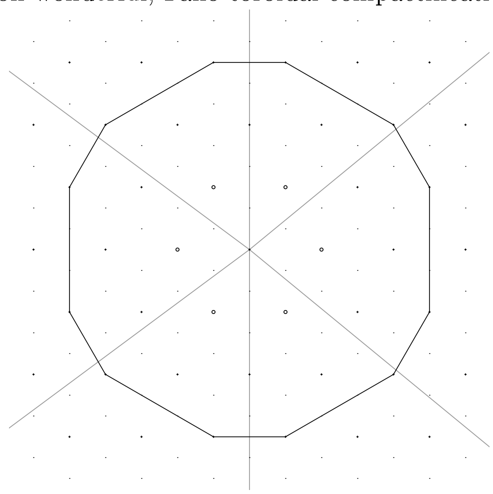
In particular, when G contains no SL_2 factor, we have $\mathrm{Aut}^0(X) \simeq G_{ad} \times G_{ad}$. This result shows that when there is no SL_2 factor, the wonderful compactification of the adjoint group is neither homogeneous nor toric. It is not homogeneous because $\mathrm{Aut}^0(X)$ leaves the boundary $X \setminus G_{ad}$ invariant. It is not toric because if Z is a toric variety, every maximal torus of $\mathrm{Aut}^0(Z)$ is of the dimension of Z. Here, remark that $\dim(G) = \mathrm{rk}(G) + \mathrm{Card}(\Phi) \geq 3\mathrm{rk}(G)$ but the maximal torus of $G_{ad} \times G_{ad}$ is of dimension $2\mathrm{rk}(G)$, so X cannot be toric.

In the case of the wonderful compactification X of $\mathrm{Sp}_{2n}(\mathbb{C})$, which is not adjoint, Pezzini proved in [Pez09] that the connected automorphism group $\mathrm{Aut}^0(X)$ is the image of $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ (He studied in fact the automorphism groups of all wonderful varieties). The image of $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ in $\mathrm{Aut}^0(X)$ is furthermore the quotient of $\mathrm{Sp}_{2n}(\mathbb{C}) \times \mathrm{Sp}_{2n}(\mathbb{C})$ by the center of $\mathrm{Sp}_{2n}(\mathbb{C})$, embedded antidiagonally, and is semisimple.

4.5 Further examples of Fano group compactifications

Let us give the polytopes of some examples of smooth Fano group compactifications that are not wonderful. A classification of such manifolds when the

Figure 4.6: Non wonderful, Fano toroidal compactification of $\mathrm{PSL}_3(\mathbb{C})$



rank is less than three can be found in [Ruz12]. In particular if we focus on toroidal compactifications of simple groups, we see in [Ruz12, Table 7] that we are only missing two such manifolds. The first is the blow up of the wonderful compactification of PGL_3 at the closed orbit whose polytope is represented in Figure 4.6, and the second is the blow up of the wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$ at the closed orbit whose polytope is represented in Figure 4.7.

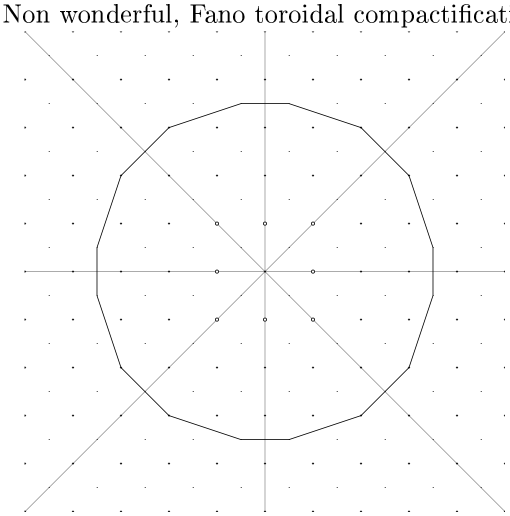
To obtain the polytopes for these examples, which are not wonderful, one can use the general description of the anticanonical divisor given by Brion in [Bri89]. In the case of a toroidal compactification, the description is simplified by Ruzzi (see [Ruz12, page 246] or his PhD thesis [Ruz]). It turns out that the support function v of the polytope of $-K_X$ can be described as $v = v_G + v_Z$ where v_G is defined by $v(x) = 2\rho(x)$ on the positive Weyl chamber and is W-invariant, and v_Z is defined as $-g_{-K_Z}(-x)$ where Z is the toric subvariety in X and g_{-K_Z} is the support function of the line bundle $-K_Z$ on Z.

For both of these manifolds, the connected group of automorphisms $\mathrm{Aut}^0(X)$ is the image of $G \times G$. It is clear that it contains this image, and we have seen that for the corresponding wonderful compactifications, the connected automorphism group is precisely this image. But since our manifolds are blowups of these wonderful manifolds, Blanchard's lemma ([Bla56, Proposition I.1], see also [BSU13, Proposition 4.2.1]) gives an inclusion of the connected automorphism groups of the blow ups in the connected automorphism groups of the wonderful ones. So we get our statement.

4.6 Hermitian metrics on line bundles

We will describe in this section how to see the potential of a $K \times K$-invariant (singular) hermitian metric h on L as a W-invariant function on $\mathfrak{a}$.

Figure 4.7: Non wonderful, Fano toroidal compactification of $\mathrm{Sp}_4(\mathbb{C})$



Applying this to non-negatively curved singular metrics, we get a correspondence between the $K\times K$-invariant singular, non-negatively curved metrics on L and W-invariant convex functions on $\mathfrak{a}$ satisfying asymptotic conditions. These conditions are given in terms of the polytope P, and are the conditions corresponding to toric metrics on the restriction of $L|_Z$ to the toric submanifold Z of X.

4.6.1 Potentials and quasi psh functions

Local potentials

Let X be a compact Kähler manifold and L a line bundle on X. A *hermitian metric* h on L is the data, for each $x \in X$ of a hermitian form on the fiber L_x over X. Given a local trivialization of L, say s, on an open subset $U \subset X$, we have a choice of basis for each space L_x, $x \in U$. So the data of a hermitian form on L_x in this basis is just a complex number $|s(x)|_h^2$, the norm of $s(x)$ with respect to the hermitian form.

We can define a function φ on U, by

$$x \mapsto -\ln(|s(x)|_h^2)$$

that we call the local potential of h with respect to s.

A hermitian metric h is determined by all its local potentials. The hermitian metric is called *smooth* (resp. *continuous*) if all of its local potentials are smooth (resp. continuous). We will consider also singular hermitian metrics which are those for which the local potentials are L^1_{loc}. Finally a hermitian metric h on L is said to be *locally bounded* if all its local potentials on sufficiently small open sets are bounded.

To a smooth hermitian metric on L is associated a (1,1)-form ω_h, called its *curvature*. One way to define it is locally: if φ is a local potential of h on U, then it is also a local $\partial\overline{\partial}$ potential of ω i.e. $\omega_h = i\partial\overline{\partial}\varphi$ on U. Furthermore, ω_h lies in the first chern class $c_1(L)$. The curvature is still well defined as a current for singular hermitian metrics thanks to the assumption on the potentials.

Global potential

There is another notion of potential for a hermitian metric h, this time global, given a reference metric h_0. Define the potential of h with respect to h_0 to be the function ψ such that for $\xi \in L_x$,

$$|\xi|_h^2 = e^{-\psi(x)}|\xi|_{h_0}^2.$$

Remark that both curvature forms lie in $c_1(L)$ so by the $\partial\overline{\partial}$-lemma there exists a function ψ such that

$$\omega_{h_0} = i\partial\overline{\partial}\psi + \omega_h.$$

The potential of h with respect to h_0 is such a function.

Positivity

A smooth hermitian metric on L is said to have *positive curvature* if its curvature ω_h is a Kähler metric. Remark that the existence of such a metric is equivalent to the ampleness of the line bundle L. More generally, a singular metric h is said to have *non-negative curvature* if $\omega_h \geq 0$ as a current.

At the level of local potentials this translates in the following way: a singular hermitian metric has non-negative curvature if its local potentials are psh functions, and it is smooth and has positive curvature if and only if its local potentials are smooth and strictly psh functions.

Let h_0 be a continuous, non negatively curved metric on L, and let ω_{h_0} be its curvature current. Define the *ω_{h_0}-psh functions* on X as the upper semicontinuous functions φ on X such that $\omega + i\partial\overline{\partial}\varphi \geq 0$.

The ω_{h_0}-psh functions parametrize all non negatively curved metrics on L, as the potentials of such metrics with respect to h_0.

4.6.2 Convex potential

Let G be a reductive complex group. Let X be a $G \times G$-equivariant smooth compactification of G. Let L be a $G \times G$-linearized ample line bundle on X. Denote by P the associated polytope.

We identify G with its open dense orbit in X, and let s_0 be a fixed left-G-equivariant trivialization of $L|_G$ given by Proposition 4.2. Denote by ψ the potential of h on G with respect to s_0:

$$\psi(z) := -\ln(|s_0(z)|_h^2).$$

Remark that, restricted to T, ψ is the potential of the hermitian metric on $L|_T$ induced by h with respect to the restriction of s_0.

Proposition 4.23. *Assume that h is $K \times K$-invariant, then ψ is also $K \times K$-invariant.*

Proof. Let $k_1, k_2 \in K$ and $z \in G$. By using the trivialization s_0, we can identify $L|_G$ with $G \times \mathbb{C}$, with the action of $(g_1, g_2) \in G \times G$ sending (z, t) to $(g_1 z g_2, \chi(g_2)t)$ for some character χ of G. Then

$$\begin{aligned}\psi(k_1 z k_2) &= -\ln(|s_0(k_1 z k_2)|_h^2) \\ &= -\ln(|(k_1 z k_2, 1)|_h^2) \\ &= -\ln(|\chi(k_2)|^{-2}|(k_1 z k_2, \chi(k_2))|_h^2) \\ &= -\ln(|\chi(k_2)|^{-2}|(k_1, k_2) \cdot (z, 1)|_h^2)\end{aligned}$$

Remark that since K is compact, $|\chi(k_2)| = 1$, and by $K \times K$-invariance of h, we get

$$\psi(k_1 z k_2) = -\ln(|s_0(z)|_h^2) = \psi(z).$$

□

Let $\varphi(x) = \psi(\exp(x))$ be the function induced by ψ on the Lie algebra $\mathfrak{a}$. Recall from Proposition 3.16 that ψ is completely determined by φ, which is W-invariant.

Suppose now that h is non negatively curved. Then ψ is psh, and so φ is convex by Proposition 3.17.

Definition 4.24. We will call φ the *convex potential* of h.

4.6.3 Asymptotic behavior of the convex potential

A special metric

Before stating the theorem giving the asymptotic behavior of non negatively curved metrics on L, we need to introduce a special continuous metric, that we will denote by h_L. This will replace the Batyrev-Tschinkel metric (see for example Appendix A or [Mai00]) defined in the toric case. In fact it is constructed from this metric on the toric submanifold.

Recall that we denote by P the polytope of the polarization (X, L), and that it is also the polytope associated to the polarized toric manifold $(Z, L|_Z)$. Let g_P be the support function associated to the line bundle $L|_Z$.

Then there exists a continuous hermitian metric h_P on $L|_Z$, toric, W-invariant and non negatively curved, called the Batyrev-Tschinkel metric and whose convex potential is the function $f_P : x \mapsto -2g_P(-x)$. Remark that if we define the support function v of a polytope Q as

$$v(x) = \sup\{\langle x, q\rangle \,;\, q \in Q\}$$

then f_P is the support function of the polytope $2P$.

Remark 4.25. We consider here the support function of $2P$ because in the definition of the potential we took $||s_0||_h^2$ and not just $||s_0||_h$.

We can extend h_P to a continuous hermitian metric h_L on L, since h_P is W-invariant. Indeed, the stabilizer in $K \times K$ of a point x of Z acts linearly on the fiber L_x, which is a complex line, and so the metric h_P is invariant under this action. The convex potential of h_L is still f_P.

Asymptotic behavior

Theorem 4.26. *The singular hermitian $K \times K$-invariant metrics h with non negative current curvature are in bijection with the convex W-invariant functions $\varphi : \mathfrak{a} \longrightarrow \mathbb{R}$ satisfying the condition that there exists a $C_1 \in \mathbb{R}$ such that*

$$\varphi(x) \leq f_P(x) + C_1$$

on $\mathfrak{a}$, and φ is then the convex potential of h. Furthermore, h is locally bounded if and only if there exists in addition a constant C_2 such that

$$f_P(x) + C_2 \leq \varphi(x) \leq f_P(x) + C_1.$$

Proof. Let h be a singular hermitian $K \times K$-invariant metric with non negative current curvature on L. Let φ be its convex potential. Recall that h_L denotes the metric constructed above, and let ω_L be the curvature current of h_L. Write v the potential of h with respect to h_L. It is an ω_L-psh metric on X. In particular, v is bounded from above on X.

Denote by u the function on $\mathfrak{a}$ associated to the $K \times K$-invariant function $v|_G$. Then we see that the function $\varphi - f_L$ is equal to u and thus bounded from above.

If furthermore h is locally bounded then since h_L is also locally bounded, the function v is bounded on X. So $u = \varphi - f_L$ is bounded on $\mathfrak{a}$.

Conversely, let φ be a convex W-invariant function such that $\varphi(x) \leq f_P(x) + C$. We choose any reference metric h_0 on L that is smooth, positively curved and $K \times K$-invariant. Then by the first direction there exist constants C_1 and C_2 such that if φ_0 is the potential of h_0 we have

$$f_P(x) + C_2 \leq \varphi_0(x) \leq f_P(x) + C_1.$$

Let ω_0 be the curvature form of h_0.

Consider the function $u := \varphi - \varphi_0$. It will be enough to show that the function v on G corresponding to u extends to an ω_0-psh function on X.

First remark that $v = \psi - \psi_0$, and by Proposition 3.17, ψ is psh on G. The assumption on φ implies that u, and thus v, are bounded from above. Indeed, we have

$$u = \varphi - \varphi_0 \leq f_P + C - \varphi_0 \leq C - C_2.$$

A classical result on psh functions is that a psh function extends over an analytic subset if and only if it is locally bounded above. Here applying that with v allows to extend v to an ω_0-psh function on X. The corresponding singular hermitian metric h has non negative curvature, is $K \times K$-invariant, and has convex potential φ.

For locally bounded metrics, one just needs to use the refinement that if a psh function is locally bounded then it extends to a bounded psh function. $\square$

Smooth metrics

In the case of polarized toric manifolds, Guillemin [Gui94] found a necessary and sufficient condition for a smooth strictly convex function to be the convex potential of a smooth positively curved toric hermitian metric. This condition is that the Legendre transform u of the convex function is of the form

$$u(p) = \sum_i l_i(p) \ln(l_i(p)) + v(p)$$

where v is a smooth function on $2P$, and the l_i are the linear forms defining $2P$. Alexeev and Katzarkov state that the condition still holds on smooth polarized group compactifications [AK05, Proposition 3.2]. We will not use this condition here.

There are simpler consequences of a metric being smooth, which we will use thoroughly in the following and in Chapter 6.

Proposition 4.27. *Let h be a smooth $K \times K$-invariant hermitian metric with positive curvature on L, and let φ be its convex potential. Then the gradient $\nabla\varphi$ of φ defines a diffeomorphism from $\mathfrak{a}$ to the interior of $2P$, identifying $\mathfrak{a}$ with $\mathfrak{a}^*$ by the scalar product $\langle\cdot,\cdot\rangle$. Furthermore, the restriction of $\nabla\varphi$ to $\mathfrak{a}^+$ is a diffeomorphism to the interior of $2P^+$.*

Proof. Since h is smooth and positively curved, φ is a smooth and strictly convex function on $\mathfrak{a}$. So $\nabla\varphi$ is a diffeomorphism. It remains to determine the image. The smoothness of h implies that it is locally bounded. So by Theorem 4.26, we have

$$f_P(x) + C_1 \leq \varphi(x) \leq f_P(x) + C_2$$

where f_P is the support function of the polytope $2P$. This implies that $\nabla\varphi(\mathfrak{a}) = \mathrm{Int}(2P)$. By W-invariance, we also have $\nabla\varphi(\mathfrak{a}^+) = \mathrm{Int}(2P^+)$. $\square$

4.7 Volume forms and the Duistermaat-Heckman measure

4.7.1 Moment map and Duistermaat-Heckman measure

Let (X, L) be a smooth polarized compactification of G, corresponding to the polytope P. Let ω be a $K \times K$-invariant Kähler form in $c_1(L)$.

Consider the *moment map* μ of (X, ω) with respect to the action of $K \times K$. The intersection of the image of μ with the positive Weyl chamber in $(\mathfrak{a} \oplus \mathfrak{a})^*$

(regarded as a subspace of $(\mathfrak{k} \oplus \mathfrak{k})^*$) is a convex polytope, called the (Kirwan) *moment polytope*.

It follows from the work of Brion [Bri87] that this moment polytope can be identified with P^+ the intersection of $P \subset M_{\mathbb{R}}$ with the positive Weyl chamber in $\mathfrak{a}^*$, where we identify $\mathfrak{a}$ with its antidiagonal embedding in $\mathfrak{a} \oplus \mathfrak{a}$.

The *Duistermaat-Heckman* measure $d\sigma$ is the pushforward of the Liouville measure $\omega^n/n!$ under the moment map $\mu : X \to P$.

Brion [Bri89] found an explicit expression for the Duistermaat-Heckman measure in this situation. Let dq be the Lebesgue measure on P normalized to give unit volume to the fundamental domain of the lattice in P. Let Ψ denote the root system of $G \times G$, which is the disjoint union of two copies of Φ, and Ψ^+ a choice of positive roots, compatible with the choice of Φ^+. Let also $\rho_{G\times G}$ denote the half sum of the positive roots of $G \times G$. Then the density of $d\sigma$ with respect to dq is

$$\nu_{DH}(q) = \prod_{\beta \in \Psi^+} \frac{(\beta, q)}{(\beta, \rho_{G\times G})}.$$

4.7.2 Degree of an ample line bundle

Proposition 4.28. *Let (X, L) be a smooth polarized compactification of G, corresponding to the polytope P. Then*

$$\deg(L) = C \int_{2P^+} \prod_{\alpha \in \Phi^+} (\alpha(p))^2 dp$$

for some constant C depending only on the group G. Furthermore, if u is the convex potential of a smooth positively curved $K \times K$-invariant metric on L, then

$$\deg(L) = C \int_{\mathfrak{a}^+} \prod_{\alpha \in \Phi^+} (\alpha(\nabla u(a)))^2 \mathrm{MA}_{\mathbb{R}}(u)(a) da.$$

Proof. Let h be a smooth positively curved $K \times K$-invariant hermitian metric on L, with curvature the Kähler form ω. Let s be a $G \times \{e\}$-equivariant section of L, and φ the potential of h with respect to s. We thus have $\omega = i\partial\bar{\partial}\varphi$ on G.

Let dg denote the Haar measure obtained on G by the choice of the basis of $\mathfrak{g}$ made in Section 3.1.6. If $z_1, \dots, z_n$ denote the local complex coordinates in which we computed the complex Hessian, locally we can write

$$\begin{aligned}\omega^n &= i^n \mathrm{MA}_{\mathbb{C}}(\varphi) dz_1 \wedge \dots \wedge d\overline{z_n} \\ &= \mathrm{MA}_{\mathbb{C}}(\varphi) dg\end{aligned}$$

This is in fact well defined on G because dg and ϕ are.

Let u denote the convex potential of h, defined by $u(a) = \varphi(\exp(a))$ for $a \in \mathfrak{a}$.

Then

$$\begin{aligned}\deg(L) &= \int_X \omega^n \\ &= \int_X \mathrm{MA}_{\mathbb{C}}(\varphi) dg \\ &= \int_G \mathrm{MA}_{\mathbb{C}}(\varphi) dg\end{aligned}$$

by KAK-integration, this is, for a constant C depending only on G and the choice of Haar measure,

$$= C \int_{\mathfrak{a}^+} \mathrm{MA}_{\mathbb{C}}(\varphi)(\exp(a)) J(a) da$$

from the expression of the complex Monge-Ampère we obtain that this is

$$= C \int_{\mathfrak{a}^+} \prod_{\alpha \in \Phi^+} \alpha(\nabla u(a))^2 \mathrm{MA}_{\mathbb{R}}(u)(a) da$$

We use the Legendre transform to transport this integral to an integral on P^+. Simply put, since u is smooth and strictly convex, we can use the variable change $p = \nabla u(a)$. Then it is clear that $dp = \mathrm{MA}_{\mathbb{R}}(u)(a) da$.

The image by ∇u of $\mathfrak{a}^+$ is the interior of $2P^+$, by Proposition 4.27 and identifying $\mathfrak{a}$ with $\mathfrak{a}^*$ by the scalar product $\langle \cdot, \cdot \rangle$, so applying the Legendre transform yields

$$\int_X \omega^n = C \int_{2P^+} \prod_{\alpha \in \Phi^+} \alpha(p)^2 dp.$$

□

Remark 4.29. This is in fact, up to a multiplicative constant, the integral with respect to the Duistermaat-Heckman measure. Indeed, we have

$$\nu_{DH}(q) = \prod_{\beta \in \Psi^+} \frac{(\beta, q)}{(\beta, \rho_{G \times G})}$$

and for $q = (p, -p)$ in $\mathfrak{a} \oplus \mathfrak{a}$, so if $\beta = (\alpha, 0)$ or $\beta = (0, -\alpha)$ for $\alpha \in \Phi$, we have

$$(\beta, q) = 2\alpha(p).$$

Thus for some constant C,

$$\nu_{DH}(p, -p) = C \prod_{\alpha \in \Phi^+} \alpha(p)^2 dp.$$

Remark 4.30. The same proof would give, for g any continuous function on P^+,

$$\int_{P^+} g(p) \prod_{\alpha \in \Phi^+} (\alpha(p))^2 dp = \int_{\mathfrak{a}^+} g(\nabla u(a)) \prod_{\alpha \in \Phi^+} (\alpha(\nabla u(a)))^2 \mathrm{MA}_{\mathbb{R}}(u)(a) da.$$

Remark 4.31. The constants that appear in our statements above could be determined explicitly by studying precisely which Haar measure appears with our choice of basis in Chapter 3. But for our purposes we will never need to determine these constants explicitly.

Chapter 5

Alpha invariants of polarized group compactifications

In this chapter we compute the alpha invariant of any linearized ample line bundle L on a group compactification X of G with respect to the action of $K \times K$ a maximal compact subgroup of $G \times G$. This is done by computing the log canonical thresholds of non negatively curved singular $K \times K$-invariant hermitian metric on L in terms of a convex body associated to it, that we call the Newton body of the metric.

To this end we first translate the log canonical threshold of a metric to an integrability condition on the global potential of the metric with respect to a fixed reference metric. Then, restricting to the dense orbit G, and using the KAK integration formula, this becomes an integrability statement for convex potentials of metrics.

Starting from the analytic version of the computation of log canonical thresholds of monomial ideals, we obtain an integrability criterion in our situation, involving the Newton bodies previously mentioned. Using this criterion and the Weyl group action we obtain an expression of the alpha invariant, that is particularly simple in the case of the compactification of a semisimple group.

We then compute the alpha invariant of the anticanonical line bundle for some examples of group compactifications. The sufficient criterion of existence of Kähler-Einstein metrics in terms of alpha invariant is unfortunately never satisfied, despite the fact that, at least for most wonderful compactifications, the group $K \times K$ is a maximal compact subgroup of $\mathrm{Aut}^0(X)$.

5.1 Log canonical thresholds on compact manifolds

In this first section we consider X a compact complex manifold that is not necessarily a group compactification, and L a line bundle on X.

Definition 5.1. Let x be a point in X, and h a hermitian metric on L. The *complex singularity exponent* (or *local log canonical threshold*) of h at x, which we denote by $\mathrm{lct}(h, x)$ is the supremum of all $c > 0$ such that $e^{-c\varphi}$ is integrable with respect to Lebesgue measure in a neighborhood of x, where φ is the potential of h with respect to a trivialization s of L in a neighborhood of x:

$$\varphi(z) := -\ln(|s(z)|_h^2).$$

Remark 5.2. If h is a locally bounded metric then on a sufficiently small neighborhood of any point, the potential φ is a bounded function, so it is integrable. It means that for any such metric, $\mathrm{lct}(h, x) = \infty$ at any point x.

Definition 5.3. Let h be a hermitian metric on L, then the *log canonical threshold* of h is defined as

$$\mathrm{lct}(h) = \inf_{x \in X}(\mathrm{lct}(h, x)).$$

Proposition 5.4. *Let h be a singular hermitian metric on L, h_0 a locally bounded hermitian metric on L, and ψ the potential of h with respect to h_0. Let also dV be any smooth volume form on X. Then we have*

$$\mathrm{lct}(h) = \sup\left\{c > 0; \int_X e^{-c\psi} dV < \infty\right\}.$$

Proof. Let x be any point in X, and s a trivialization of L on a neighborhood U of x. Up to shrinking U, we can assume that the local potential φ_0 of h_0 with respect to s is bounded.

Let φ be the local potential of h with respect to s and ψ the potential of h with respect to h_0. Then by definition of ψ, we have $\psi = \varphi - \varphi_0$ on U, and since φ_0 is bounded, the integrability of $e^{-c\varphi}$ with respect to Lebesgue measure on a neighborhood of x is equivalent to the integrability of $e^{-c\psi}$ on the same neighborhood.

Furthermore, in the neighborhood of any point x in X, the integrability with respect to Lebesgue measure is equivalent to integrability with respect to a smooth volume form.

The function ψ is defined everywhere on X, $e^{-c\psi}$ is positive, and X is compact, so $e^{-c\psi}$ is integrable with respect to dV in the neighborhood of any point in X if and only if $\int_X e^{-c\psi} dV < \infty$.

Take $0 < c < \mathrm{lct}(h)$, then $c < \mathrm{lct}(h, x)$ for all $x \in X$, so $\int_X e^{-c\psi} dV < \infty$. This means that

$$\mathrm{lct}(h) \leq \sup\left\{c > 0; \int_X e^{-c\psi} dV < \infty\right\}.$$

Conversely, if $c > \mathrm{lct}(h)$ then there exists $x \in X$ such that $c > \mathrm{lct}(h, x)$ but then $\int_X e^{-c\psi} dV = \infty$, so $c \geq \sup\left\{c > 0; \int_X e^{-c\psi} dV < \infty\right\}$. Taking the infimum gives the other inequality:

$$\mathrm{lct}(h) \geq \sup\left\{c > 0; \int_X e^{-c\psi} dV < \infty\right\}.$$

This proves the proposition. $\square$

5.2 Newton body of a hermitian metric

In this section we introduce a convex body associated to any non negatively curved singular $K \times K$-invariant hermitian metric h on an ample linearized bundle L on a group compactification X. We first define a convex set associated to any function, which is a natural set to consider in the case of convex functions. Applying this construction to the convex potential of a hermitian metric yields a convex body that is contained in the polytope of L, that will be used to compute the log canonical threshold of h.

5.2.1 Newton set of a function

Definition 5.5. Let f be a function $\mathfrak{a} \to \mathbb{R}$, and σ a convex cone in $\mathfrak{a}$. We call *Newton set* of f the following set in $\mathfrak{a}^*$

$$N_\sigma(f) = \{m \in \mathfrak{a}^*; \exists C, \forall x \in \sigma, f(x) - m(x) \geq C\}.$$

For any function f and any convex cone σ, the Newton set of f is clearly convex.

Recall the definition of the dual cone $\sigma^\vee$ of σ:

$$\sigma^\vee = \{m \in \mathfrak{a}^*; m(x) > 0 \ \forall x \in \sigma\}.$$

The Newton set $N_\sigma(f)$ is by definition stable under addition of an element of the closure of the opposite of the dual cone $\overline{\sigma^\vee} \subset \mathfrak{a}^*$. We write this also $N_\sigma(f) = N_\sigma(f) + (-\overline{\sigma^\vee})$ where the plus sign means the Minkowki sum.

Example 5.6. Let f be the affine function $f(x) = m(x) + c$ where $m \in \mathfrak{a}^*$ and c is a constant. Then $N_\sigma(f) = m + (-\overline{\sigma^\vee})$.

Let us record the following elementary properties of Newton sets.

Proposition 5.7. *Let f and g be two functions on $\mathfrak{a}$ and $c \in \mathbb{R}$. Then*

- $N_\sigma(cf) = cN_\sigma(f)$
- $N_\sigma(f + c) = N_\sigma(f)$
- *if $f \leq g$ then $N_\sigma(f) \leq N_\sigma(g)$.*

In particular, if for some constants c_1 and c_2,

$$g + c_1 \leq f \leq g + c_2$$

on σ, then $N_\sigma(f) = N_\sigma(g)$.

If now the cone is changing instead of the function, we have again some easy properties of the Newton set.

Proposition 5.8. *Let $f : \mathfrak{a} \to \mathbb{R}$ be any function.*

- *If σ is covered by other cones σ_i* i.e. *$\overline{\sigma} = \cup \overline{\sigma_i}$, then $N_\sigma(f) = \bigcap N_{\sigma_i}(f)$.*
- *If $\sigma_1 \subset \sigma_2$ then $N_{\sigma_2}(f) \subset N_{\sigma_1}(f)$.*

Let us now prove a less trivial result which will be used in the computation of log canonical thresholds.

Proposition 5.9. *Let $v : \mathfrak{a} \to \mathbb{R}$ be a* convex, piecewise linear *function along a decomposition $\overline{\sigma} = \cup \overline{\sigma_i}$ of a cone in cones of full dimension. Denote by v_{σ_i} the element of $\mathfrak{a}^*$ such that $v(x) = v_{\sigma_i}(x)$ on σ_i. Then $N_\sigma(v) = \mathrm{Conv}\{v_{\sigma_i}\} + (-\overline{\sigma^\vee})$.*

Proof. First, it is clear that $N_{\sigma_i}(v) = v_{\sigma_i} + (-\overline{\sigma_i^\vee})$ for all i, and thus, that $N_\sigma(v) = \bigcap_i v_{\sigma_i} + (-\overline{\sigma_i^\vee})$.

In particular, we have the easy inclusion $N_\sigma(v) \subset \mathrm{Conv}\{v_{\sigma_i}\} + (-\overline{\sigma^\vee})$.

To prove the other inclusion, it is enough to show that for all i, j, $v_{\sigma_i} \in v_{\sigma_j} + (-\overline{\sigma_j^\vee})$.

We use the convexity of v. Let $x \in \mathrm{Int}(\sigma_i)$ and $y \in \mathrm{Int}(\sigma_j)$. We have, for any $0 \leq t \leq 1$,

$$v(ty + (1-t)x) \leq tv_{\sigma_j}(y) + (1-t)v_{\sigma_i}(x).$$

When t is close to 0, $ty + (1-t)x$ is still in σ_i, so we get

$$tv_{\sigma_i}(y) + (1-t)v_{\sigma_i}(x) \leq tv_{\sigma_j}(y) + (1-t)v_{\sigma_i}(x).$$

This implies that

$$(v_{\sigma_j} - v_{\sigma_i})(y) \geq 0.$$

This is true for all $y \in \mathrm{Int}(\sigma_j)$, so for all $y \in \sigma_j$, so this means that $v_{\sigma_j} - v_{\sigma_i} \in \sigma_j^\vee$, or in another order $v_{\sigma_i} \in v_{\sigma_j} + (-\overline{\sigma_j^\vee})$. $\square$

5.2.2 Newton set of convex functions

For this paragraph only, we will allow convex functions to take the value $+\infty$. If f is such a function we define its *domain* by

$$\mathrm{dom}(f) := \{x \in \mathfrak{a}; f(x) < \infty\}.$$

We impose however that all functions considered have a non empty domain. In the rest of the text, we assume $\mathrm{dom}(f) = \mathfrak{a}$.

The first remark to be made is that the Newton set of a function f on the whole of $\mathfrak{a}$ is the domain of its Legendre-Fenchel transform (or convex conjugate) f^* defined, for $m \in \mathfrak{a}^*$, by

$$f^*(m) := \sup\{m(x) - f(x); x \in \mathfrak{a}\}.$$

Let σ be a convex cone, and define the convex function δ_σ as the indicator function of σ, *i.e.* $\delta_\sigma(x) = 0$ if $x \in \sigma$ and $\delta_\sigma(x) = \infty$ otherwise. Then it is not hard to check that $N_\sigma(f) = N_{\mathfrak{a}}(f + \delta_\sigma)$. In other words $N_\sigma(f)$ is the domain of the convex conjugate of $f + \delta_\sigma$.

We will recall a classical result on convex functions, which allows to express the Newton set of a sum as the Minkowski sum of the Newton sets of the summands. First recall the definition of infimal convolution:

Definition 5.10. Let f and g be two convex function. The *infimal convolution* of f and g is the function $f\Box g$ defined, for $x \in \mathfrak{a}$, by

$$f\Box g(x) = \inf\{f(x-y) + g(y); y \in \mathfrak{a}\}.$$

Theorem 5.11. *[Roc97, Theorem 16.4] Let f and g be two convex functions on $\mathfrak{a}$, such that the relative interiors of the domains of f and g have a point in common. Then*

$$(f+g)^*(m) = f^*\Box g^*.$$

Proposition 5.12. *Let σ be a convex cone, and f a convex function with* $\mathrm{dom}(f) = \mathfrak{a}$*, then*

$$N_\sigma(f) = N_{\mathfrak{a}}(f) + (-\overline{\sigma^\vee}).$$

Proof. We have seen that $N_\sigma(f)$ is the domain of the convex conjugate of $f+\delta_\sigma$, but by Theorem 5.11, this is also the domain of the function $f^*\Box\delta_\sigma^*$. We can apply the Theorem because the intersection of the domains of f and δ_σ is σ.

The domain of an infimal convolution is the Minkowski sum of the domains of the two functions involved, so we just need to compute the domain of δ_σ^*. By definition we check that this is $-\sigma^\vee$, and obtain the statement. □

Proposition 5.13. *Let f and g be two convex functions, both with domain $\mathfrak{a}$, and σ a convex cone. Then $N_\sigma(f+g) = N_\sigma(f) + N_\sigma(g)$.*

Proof. We have, by the previous proposition,

$$N_\sigma(f+g) = N_{\mathfrak{a}}(f+g) + (-\overline{\sigma^\vee}).$$

But by the same proof,

$$N_{\mathfrak{a}}(f+g) = N_{\mathfrak{a}}(f) + N_{\mathfrak{a}}(g),$$

so

$$\begin{aligned} N_\sigma(f+g) &= N_{\mathfrak{a}}(f) + N_{\mathfrak{a}}(g) + (-\overline{\sigma^\vee}) \\ &= N_\sigma(f) + N_\sigma(g) \end{aligned}$$

□

5.2.3 Newton body of a metric

Let X be a compactification of G, polarized by L. Let h be a $K\times K$-invariant hermitian metric with non negative curvature on L, and φ its convex potential with respect to a fixed left-invariant trivialization of L on G, which is a function on $\mathfrak{a}$.

Definition 5.14. We will call *Newton body* of h the set $N(h) := N_{\mathfrak{a}}(\varphi)$.

Let P_L be the polytope corresponding to the polarization L.

Example 5.15. Let h_L be the metric constructed in Section 4.6.3. Its convex potential is the support function of $2P_L$, so $N(h_L) = 2P_L$, which can be checked by Proposition 5.9. Remark that the convex potential of h_L is piecewise linear with respect to the opposite of the fan of the toric subvariety.

Proposition 5.16. *The Newton body of h is stable under the action of the Weyl group W.*

Proof. Let φ be the convex potential of h, and let $m \in \mathfrak{a}^*$. Suppose that

$$\varphi(x) - m(x) \geq C$$

for some constant C and for all $x \in \mathfrak{a}$. Let $w \in W$. By W-invariance of φ, the inequality is equivalent to

$$\begin{aligned} C &\leq \varphi(w \cdot x) - m(x) \\ &\leq \varphi(w \cdot x) - w^{-1} \cdot m(w \cdot x) \end{aligned}$$

Since w induces a bijection of $\mathfrak{a}$, we get that for all $w \in W$, $m \in N(h)$ if and only if $w \cdot m \in N(h)$, which means that $N(h)$ is W-invariant. $\square$

Proposition 5.17. *Let h be a $K \times K$-invariant hermitian metric with non negative curvature on L. Then $N(h) \subset 2P_L$. If in addition h is locally bounded, then $N(h) = 2P_L$.*

Proof. Denote by φ_L the convex potential of the metric constructed in Section 4.6.3. Recall from the same section that the convex potential φ of a $K \times K$-invariant hermitian metric h with non negative curvature on L satisfies

$$\varphi \leq \varphi_L + C_2$$

on $\mathfrak{a}$ for some constant C_2, and that if h is locally bounded then we have in addition

$$\varphi_L + C_1 \leq \varphi$$

for some constant C_1.

Now the result easily follows from Proposition 5.7 and Example 5.15. $\square$

5.3 Integrability criterions

5.3.1 Integrability criterion on a cone

We will use the following proposition, proved by Guenancia in [Gue12]. It is an analytic proof and generalization of the computation by Howald of the log canonical thresholds of monomial ideals. The statement given here is a slightly different than the statement in [Gue12], but is in fact equivalent (see Appendix A for details).

Proposition 5.18. *[Gue12] Let f be a convex function on $\mathfrak{a}$. Assume that σ is a smooth polyhedral cone in $\mathfrak{a} = N_{\mathbb{R}}$. Then e^{-f} is integrable on a translate (equivalently on all translates) of σ if and only if 0 is in the interior of the Newton body of f: $0 \in \mathrm{Int}(N_\sigma(f))$.*

5.3.2 Integrability with respect to J

Fix G a reductive group, let Φ be its root system, Φ^+ a choice of positive roots. Recall that J is the function defined on $\mathfrak{a}$ by

$$J(x) = \prod_{\alpha \in \Phi^+} \sinh^2(\alpha(x)).$$

The half sum of positive roots is denoted by ρ.

We want to prove the following integrability criterion, with respect to the measure $J(x)dx$.

Proposition 5.19. *Assume that $\overline{\mathfrak{a}^+} = \bigcup_i \overline{\sigma_i}$ where each σ_i is a smooth polyhedral cone of full dimension r. Let l be a function on $\mathfrak{a}$, convex on each cone σ_i. Then*

$$\int_{\mathfrak{a}^+} e^{-l(x)} J(x) dx < +\infty$$

if and only if $4\rho \in \mathrm{Int}(N_{\mathfrak{a}^+}(l))$.

Lemma 5.20. *Let σ be a smooth full dimensional polyhedral cone in $\mathfrak{a}^+$, l be a convex function on $\mathfrak{a}$, then the following are equivalent:*

- $\int_\sigma e^{-l(x)} J(x) dx < \infty$;
- $\int_\sigma e^{-l(x)+4\rho(x)} dx < \infty$;
- $4\rho \in \mathrm{Int}(N_\sigma(l))$.

Proof. Writing

$$\sinh(\alpha(x)) = \frac{e^{\alpha(x)} - e^{-\alpha(x)}}{2} = \frac{1}{2} e^{\alpha(x)}(1 - e^{-2\alpha(x)}),$$

we get that

$$J(x) = \frac{1}{2^{2\mathrm{Card}(\Phi^+)}} e^{2\sum_{\alpha\in\Phi^+} \alpha(x)} \prod_{\alpha \in \Phi^+} (1 - e^{-2\alpha(x)})^2.$$

For any $x \in \mathfrak{a}^+$ and $\alpha \in \Phi^+$, $\alpha(x) > 0$, so $0 \leq e^{-2\alpha(x)} < 1$. This implies $0 < \prod_{\alpha\in\Phi^+}(1 - e^{-2\alpha(x)})^2 \leq 1$, so

$$0 < J(x) \leq \frac{1}{2^{2\mathrm{Card}(\Phi^+)}} e^{4\rho(x)}.$$

This first inequality allows to say that if $\int_\sigma e^{-l(x)+4\rho(x)} dx < \infty$ then

$$\int_\sigma e^{-l(x)} J(x) dx < \infty.$$

Let us now prove the converse. Choose γ a point in the interior of σ. Assume that $e^{-l+4\rho}$ is not integrable on σ. Then by the usual integrability criterion (Proposition 5.18) $e^{-l+4\rho}$ is also non integrable on $\gamma + \sigma$.

But now, for $x \in \gamma + \mathfrak{a}^+$ and $\alpha \in \Phi^+$, we have $\alpha(x) \geq c = \min_{\beta\in\Phi^+}\beta(\gamma) > 0$, so $0 \leq e^{-2\alpha(x)} \leq e^{-2c} < 1$, and this implies

$$\left(\frac{1-e^{-2c}}{2}\right)^{2\mathrm{Card}(\Phi^+)} e^{4\rho(x)} \leq J(x) \leq \frac{1}{2^{2\mathrm{Card}(\Phi^+)}} e^{4\rho(x)}.$$

This gives that

$$\begin{aligned}\int_\sigma e^{-l(x)}J(x)dx &\geq \int_{\gamma+\sigma} e^{-l(x)}J(x)dx \\ &\geq \int_{\gamma+\sigma} e^{-l+4\rho}dx \\ &\geq \infty\end{aligned}$$

So we have shown the equivalence of the two first points in the lemma. By the usual criterion the second point is also equivalent to

$$0 \in \mathrm{Int}(N_\sigma(l-4\rho)) = -4\rho + \mathrm{Int}(N_\sigma(l)).$$

Letting 4ρ go to the left, we conclude the proof. □

Now we can prove the proposition, just by gluing the parts.

Proof. Just remark that since the function $e^{-l(x)}J(x)$ is positive and the cones are full dimensional, $\int_{\mathfrak{a}^+} e^{-l(x)}J(x)dx < +\infty$ if and only if $\int_{\sigma_i} e^{-l(x)}J(x)dx < +\infty$ for all i.

For each of these integrals we can use the lemma, so the necessary and sufficient condition becomes $4\rho \in \mathrm{Int}(N_{\sigma_i}(l))$ for all i, or equivalently $4\rho \in \mathrm{Int}(\bigcap_i N_{\sigma_i}(l))$.

To conclude, observe that $N_{\mathfrak{a}^+}(l) = \bigcap_i N_{\sigma_i}(l)$ by Remark 5.8. □

5.4 Log canonical thresholds on group compactifications

Let G be a reductive group. Let X be a smooth Fano $G \times G$-equivariant compactification of G. Let L be an ample line bundle on X, whose associated polytope is P. Denote by Q the polytope associated to the anticanonical bundle $-K_X$.

Let also H denote the convex hull of all images of 2ρ by the Weyl group W.

We will consider only $K \times K$-invariant metrics on L, and use the KAK integration formula that we recall here:

Proposition 5.21. *[Kna02] If f is a $K\times K$ invariant function on G, dg denotes a Haar measure on G, and dx a Lebesgue measure on $\mathfrak{a}^+$, then*

$$\int_G f(g)dg = C\int_{\mathfrak{a}^+} J(x)f(\exp(x))dx$$

for some constant $C > 0$ independent of f.

We want to prove the following

Theorem 5.22. *Let h be a $K \times K$-invariant hermitian metric with non negative curvature on L, then*

$$\text{lct}(h) = \sup\{c > 0; 2H + 2cP \subset cN(h) + 2Q\}$$

We first introduce some notations.

Let us fix s_0 a left G equivariant trivialization of L on G and s_1 a left G equivariant trivialization of $-K_X$ on G.

Let u be the convex potential of h with respect to the section s_0. Let also u_0 be the support function of P and h_0 be the corresponding metric. It has locally bounded potentials. Denote by ψ the potential of h with respect to h_0.

Since X is Fano, we can choose h_1 a smooth metric on $-K_X$ with positive curvature, and let u_1 be its convex potential with respect to s_1. This choice determines a smooth volume form on X, which writes, on G,

$$dV = e^{-u_1} dg$$

where dg is the Haar measure $s_1^{-1} \wedge \overline{s_1^{-1}}$.

Remark 5.23. In particular, the integral of this volume form is finite, so applying the KAK integration formula this means that

$$\int_{\mathfrak{a}^+} e^{-u_1} J dx < \infty$$

By the integrability criterion (Proposition 5.19), this implies that

$$4\rho \in \text{Int}(N(h_1)) = \text{Int}(2Q).$$

Another way to say that is $H \subset \text{Int}(Q)$.

Proof. Using Proposition 5.4, then restricting to the dense orbit, we get:

$$\begin{aligned}
\text{lct}(h) &= \sup\left\{c > 0; \int_X e^{-c\psi} dV < \infty\right\} \\
&= \sup\left\{c > 0; \int_G e^{-c\psi} dV < \infty\right\}
\end{aligned}$$

Since $\psi(\exp(x)) = u(x) - u_0(x)$, we can now use the KAK integration formula to write:

$$\text{lct}(h) = \sup\left\{c > 0; \int_{\mathfrak{a}^+} e^{-c(u-u_0)} e^{-u_1} J dx < \infty\right\}.$$

Then Proposition 5.19 gives:

$$\begin{aligned}
\text{lct}(h) &= \sup\left\{c > 0; 4\rho \in \text{Int}(N_{\mathfrak{a}^+}(cu - cu_0 + u_1))\right\} \\
&= \sup\{c > 0; 4\rho \in N_{\mathfrak{a}^+}(cu - cu_0 + u_1)\}
\end{aligned}$$

Let σ_i be the closures of the cones of full dimension in the fan subdivision of $\mathfrak{a}^+$ corresponding to X. Then u_0 is linear on each $-\sigma_i$. We write u_0^i the corresponding element of $\mathfrak{a}^*$.

We have

$$\begin{aligned}\mathrm{lct}(h) &= \sup\{c > 0; \forall i,\ 4\rho \in N_{-\sigma_i}(cu - cu_0 + u_1)\}\\ &= \sup\{c > 0; \forall i,\ 4\rho + cu_0^i \in N_{-\sigma_i}(cu + u_1)\}\end{aligned}$$

Recall from Proposition 5.9 that $P = N_{\mathfrak{a}}(u_0) \subset u_0^i + \sigma_i^\vee$, so that

$$\begin{aligned}\mathrm{lct}(h) &= \sup\{c > 0; \forall i,\ 4\rho + cP \in N_{-\sigma_i}(cu + u_1)\}\\ &= \sup\{c > 0; 4\rho + cP \in N_{\mathfrak{a}^+}(cu + u_1)\}\\ &= \sup\{c > 0; 2H + 2cP \subset N_{\mathfrak{a}}(cu + u_1)\}\end{aligned}$$

by W-invariance.

To conclude it remains to remark that both u and u_1 are convex, so by Proposition 5.13,

$$N_{\mathfrak{a}}(cu + u_1) = cN_{\mathfrak{a}}(u) + N_{\mathfrak{a}}(u_1) = cN(h) + 2Q.$$

□

5.5 Alpha invariant on group compactifications

Definition 5.24. Let X be a compact complex manifold, K a compact subgroup of the automorphisms group of X, and L a K-linearized line bundle on X. The *alpha invariant* of L relative to the group K, denoted by $\alpha_K(L)$ is the infimum of the log canonical thresholds of all K-invariant singular hermitian metrics on L with non negative curvature.

Before stating the main result, let us introduce two notions about convex bodies.

Let P and Q be two convex bodies in $\mathfrak{a}^*$. Recall the definition of the *Minkowski difference*:

$$Q \ominus P = \{x | x + P \subset Q\}.$$

Another expression of the Minkowski difference is the following, which shows that it is convex if Q is convex:

$$Q \ominus P = \bigcap_{p \in P} (-p + Q).$$

If P_1, P_2 and Q are three convex bodies, then $P_1 + Q \subset P_2$ if and only if $P_1 \subset P_2 \ominus Q$.

Definition 5.25. The *inradius* of Q with respect to P is the number:

$$\mathrm{inr}(P, Q) := \sup\{c \geq 0 | \exists x\ x + cP \subset Q\}.$$

The alpha invariant of an ample line bundle on a smooth Fano compactification of a semisimple group can be easily expressed in terms of the polytope associated to L as an inradius between two convex bodies. We state first the result for general reductive group compactifications, and then we will see how the statement is simplified in the case when the group is semisimple.

Theorem 5.26. *Let G be a reductive group, and X be a smooth Fano group compactification of G. Let L be an ample $G \times G$-linearized line bundle on X, whose associated polytope is P. Denote by Q the polytope associated to the anticanonical line bundle $-K_X$. Then*

$$\alpha_{K\times K}(L) = \sup\{c > 0; c(P + (-P^W)) \subset Q \ominus H\},$$

where P^W denotes the subset of W-invariant points of P.

Proof. Let h be any $K \times K$-invariant metric on L with non negative curvature. The Newton body of h is convex and W-stable. In particular it contains a W-invariant point p, for example the barycenter of the orbit of any point in $N(h)$.

Denote by h_p the $K \times K$-invariant metric on L with non negative curvature whose convex potential is the function $x \mapsto p(x)$. Then $\{p\} = N(h_p) \subset N(h)$, so by the expression of the log canonical thresholds from Theorem 5.22, $\mathrm{lct}(h) \geq \mathrm{lct}(h_p)$.

Since all such h_p for $p \in 2P^W$ define a singular hermitian metric with non-negative curvature, this remark allows to write the alpha invariant as

$$\alpha_{K\times K}(L) = \inf_{p\in 2P^W} \mathrm{lct}(h_p).$$

Now from the expression of the log canonical threshold we get

$$\begin{aligned}\mathrm{lct}(h_p) &= \sup\{c > 0; 2H + 2cP \subset cN(h_p) + 2Q\}\\ &= \sup\{c > 0; -cp + 2cP \subset 2Q \ominus 2H\}\end{aligned}$$

Then the expression of the alpha invariant further simplifies as

$$\begin{aligned}\alpha_{K\times K}(L) &= \inf_{p\in 2P^W} \sup\{c > 0; -cp + 2cP \subset 2Q \ominus 2H\}\\ &= \sup\{c > 0; \forall p \in 2P^W, -cp + 2cP \subset 2Q \ominus 2H\}\\ &= \sup\{c > 0; 2cP + (-2cP^W) \subset 2Q \ominus 2H\}\end{aligned}$$

Dividing by two yields

$$= \sup\{c > 0; c(P + (-P^W)) \subset Q \ominus H\}$$

which is the expression in the statement of the Theorem. □

Remark 5.27. In the toric case, we recover our previous computation from Appendix A:

$$\alpha_{(\mathbb{S}^1)^n}(L) = \sup\{c > 0; c(P + (-P)) \subset Q\}.$$

Corollary 5.28. *Assume that G is a semisimple group. Then*

$$\alpha_{K\times K}(L) = \mathrm{inr}(P, Q \ominus H).$$

Proof. If G is semisimple, we have $P^W = \{0\}$. In fact, the metric h_0 whose convex potential is the zero function satisfies

$$\begin{aligned}\alpha_{K\times K}(L) &= \mathrm{lct}(h_0)\\ &= \sup\{c > 0; cP \subset Q \ominus H\}.\end{aligned}$$

And this is equal to the inradius $\mathrm{inr}(P, Q \ominus H)$.

Indeed, one inequality is trivial: $\mathrm{inr}(P, Q \ominus H) \geq \alpha_{K\times K}(L)$. Conversely, assume $c \leq \mathrm{inr}(P, Q \ominus H)$, i.e. there exists an $x \in \mathfrak{a}^*$ such that

$$x + cP \subset Q \ominus H.$$

Then since P and $Q \ominus H$ are stable under W-action, we also have

$$\forall w \in W,\ w \cdot x + cP \subset Q \ominus H.$$

Convexity and the fact that the barycenter of the W-orbit of x is 0 imply that $cP \subset Q \ominus H$, so $c \leq \alpha_{K\times K}(L)$. We have thus proved the other inequality $\mathrm{inr}(P, Q \ominus H) \leq \alpha_{K\times K}(L)$. □

Remark 5.29. In the case of reductive groups, the alpha invariant is not an inradius, but we can bound it from above by an inradius:

$$\alpha_{K\times K}(L) \leq \mathrm{inr}(P + (-P)^W, Q \ominus H).$$

5.5.1 Additional symmetries

If the polytopes P and Q admit additional common symmetries, then the value of the alpha invariant can be improved. Indeed, the symmetries of Q translate to a finite subgroup O of the automorphisms group of the variety X, and if P is stable under these symmetries, then it is linearized by O. We can thus consider the alpha invariant with respect to the bigger group generated by $K \times K$ and O, that we denote K_O.

We then have, adapting the proof of Theorem 5.26,

$$\alpha_{K_O}(L) = \sup\{c > 0; c(P + (-P^{\langle W,O\rangle})) \subset Q \ominus H\}.$$

In particular, if the only fixed point under $\langle W, O\rangle$ is the origin, then just as in the semisimple case, we get

$$\alpha_{K_O}(L) = \mathrm{inr}(P, Q \ominus H).$$

5.6 Examples

For wonderful compactifications of semisimple adjoint groups, the polytope of the anticanonical line bundle Q is determined by the root system. Indeed, recall that it is the convex hull of the images by W of the weight $2\rho + \sum_{i=1}^{r} \alpha_i$ where the α_i are the simple roots of Φ^+ and 2ρ is the sum of the positive roots.

In particular, when $G = (\mathrm{PSL}^2(\mathbb{C}))^n$, for any $n \geq 1$, the simple roots are the same as the positive roots, so $Q = 2H$.

Corollary 5.30. *Let X be the wonderful compactification of $(\mathrm{PSL}_2(\mathbb{C}))^n$, then*

$$\alpha_{K\times K}(-K_X) = \frac{1}{2}.$$

Proof. Applying Corollary 5.28 gives

$$\alpha_{K\times K}(-K_X) = \mathrm{inr}(2H, H) = \frac{1}{2}.$$

□

More generally for type A_n, choosing an appropriate ordering of the simple roots $\alpha_1, \ldots, \alpha_n$, we can write the positive roots as

$$\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$$

for each pair (i, j) with $1 \leq i \leq j \leq n$. We see then that the coefficient of α_k in the sum of positive roots $\sum_{l=1}^{n} \alpha_l$ is equal to the cardinal of the set $\{(i, j); 1 \leq i \leq k \leq j \leq n\}$. This is $k(n-k+1)$. Adding the sum of simple roots, we see that the k^{th}-coordinate of the vertex defining the polytope of the wonderful compactification of $\mathrm{PSL}_{n+1}(\mathbb{C})$ in the basis of simple roots is $1 + k(n-k+1)$.

Then from our result, the alpha invariant is easily seen to be the maximum of all $c > 0$ such that for each k, $c(1+k(n-k+1)) \leq 1$. We deduce the following value for the alpha invariant.

Corollary 5.31. *Let X be the wonderful compactification of $\mathrm{PSL}_{n+1}(\mathbb{C})$, then*

$$\alpha_{K\times K}(-K_X) = \frac{1}{1 + \lceil \frac{n}{2} \rceil (\lfloor \frac{n}{2} \rfloor + 1)}.$$

Chapter 6

Existence of Kähler-Einstein metrics on group compactifications

Let X be a smooth Fano compactification of a connected reductive group G. Denote by P the polytope associated to the anticanonical polarization of X.

Recall that the root system Φ of G is the root system of the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, and 2ρ denotes the sum of positive roots, in $\mathfrak{a}^*$. Let $\langle \cdot, \cdot \rangle$ denote the scalar product on $\mathfrak{a}$ introduced in Chapter 3 (it extends the Killing form on the semisimple part). We use it to identify $\mathfrak{a}$ with $\mathfrak{a}^*$. In particular, P is identified with a polytope in $\mathfrak{a}$, and P^+ with its intersection with $\mathfrak{a}^+$ because the positive Weyl chambers in $\mathfrak{a}$ and $\mathfrak{a}^*$ correspond. Recall that in the reductive case, if $\mathfrak{a}_z$ denotes the toric part and $\mathfrak{a}_{ss}$ the semisimple part of $\mathfrak{a}$, then the positive Weyl chamber $\mathfrak{a}_{ss}^+$ is defined as usual as the open Weyl chamber generated by the fundamental weights, and $\mathfrak{a}^+$ is the product

$$\mathfrak{a}^+ = \mathfrak{a}_z \times \mathfrak{a}_{ss}^+ \subset \mathfrak{a}.$$

Let Ξ be the relative interior of the closed cone generated by the simple roots. Another definition of Ξ is:

$$\Xi = \{p \in \mathfrak{a}_{ss}; \langle p, x \rangle > 0 \ \forall x \in \mathfrak{a}_{ss}^+\}.$$

The results are the following.

Theorem 6.1. *There exists a Kähler-Einstein metric on X if and only if the barycenter $\mathrm{bar}_{DH}(P^+)$ of P^+ with respect to the Duistermaat-Heckman measure is in $2\rho + \Xi$.*

An expression of the barycenter $\mathrm{bar}_{DH}(P^+)$ is the following, where Φ^+ denotes the positive roots of the root system Φ of G:

$$\mathrm{bar}_{DH}(P^+) = \left(\int_{P^+} p \prod_{\alpha\in\Phi^+}(\alpha(p))^2 dp\right)\left(\int_{P^+}\prod_{\alpha\in\Phi^+}(\alpha(p))^2 dp\right)^{-1}.$$

Observe that when G is semisimple, the cone Ξ is the open cone generated by the simple roots of Φ. When G is not semisimple, the dimension of Ξ is strictly smaller than r. In particular, for G a torus, ρ is the origin and $\Xi = \{0\}$, so we recover the usual toric criterion. Indeed, the Duistermaat-Heckman measure then is the Lebesgue measure on $P = P^+$, so the criterion is just that the barycenter of P is the origin.

When there are no Kähler-Einstein metrics, we can see how far we can go in the continuity method, and thus we have the value of the greatest Ricci lower bound. Indeed this invariant $R(X)$ of a Fano manifold X, defined as

$$R(X) := \sup\{t; \exists\omega \in c_1(X), \mathrm{Ric}(\omega) \geq t\omega\}$$

was shown to coincide with the supremum of all times at which there exists a solution in the continuity method [Szé11].

Theorem 6.2. *Assume there are no Kähler-Einstein metrics on X and let $R(X)$ be the greatest Ricci lower bound of X. Then*

$$R(X) = \sup\left\{t<1; \frac{t}{1-t}(2\rho - \mathrm{bar}_{DH}(P^+)) + 2\rho \in (P^+ + (\overline{-\Xi}))\right\}.$$

This chapter is devoted to the proof of these results. The necessary part of the condition is obtained in Proposition 6.21 and the sufficient part in Theorem 6.30. For the greatest Ricci lower bound, we first prove that it is lower than the quantity in the theorem in Proposition 6.22, and end the proof with Theorem 6.31.

6.1 Continuity method

6.1.1 In general

Let X be a Fano manifold. Fix a reference Kähler form ω_{ref} in the class $2\pi c_1(X)$. The Kähler forms in $2\pi c_1(X)$ can all be written as $\omega_{\mathrm{ref}} + i\partial\overline{\partial}\psi$ with ψ a smooth and ω_{ref}-strictly psh function on X, i.e. such that $\omega_{\mathrm{ref}} + i\partial\overline{\partial}\psi > 0$.

The Kähler-Einstein equation $\mathrm{Ric}(\omega) = \omega$ on X translates, in terms of ω_{ref}-psh functions, as the Monge-Ampère equation

$$(\omega_{\mathrm{ref}} + i\partial\overline{\partial}\psi)^n = e^{f_{\mathrm{ref}}-\psi}\omega_{\mathrm{ref}}^n, \tag{6.1}$$

where f_{ref} is the *normalized Ricci potential* of ω_{ref} defined as the ω_{ref}-psh function that satisfies $\omega_{\mathrm{ref}} + i\partial\overline{\partial}f_{\mathrm{ref}} = \mathrm{Ric}(\omega_{\mathrm{ref}})$ and $\int_X e^{f_{\mathrm{ref}}}\omega_{\mathrm{ref}}^n = \int_X \omega_{\mathrm{ref}}^n$.

Let h_{ref} be a smooth hermitian metric on $-K_X$ with curvature form ω_{ref}. Then it determines a volume form dV on X defined in a local trivialization s of $-K_X$ by $dV = |s|^2_{h_{\mathrm{ref}}} s^{-1} \wedge \overline{s^{-1}}$. Then up to a constant, the Ricci potential f_{ref} is equal to the logarithm of the potential of dV with respect to ω_{ref}^n. We choose h_{ref} (by multiplying by a scalar) such that f_{ref} is indeed equal to that.

The following family of equations is the one used in the usual continuity method for the Kähler-Einstein equation:

$$(\omega_{\mathrm{ref}} + \partial\overline{\partial}\psi_t)^n = e^{f_{\mathrm{ref}} - t\psi_t}\omega_{\mathrm{ref}}^n. \tag{6.2}$$

To show the existence of a Kähler-Einstein metric on X, it is enough to show that the set I of $0 \leq t \leq 1$ such that this equation admits a solution is exactly $[0, 1]$.

By the work of Aubin [Aub76] and Yau [Yau78], $0 \in I$, and I is open. Furthermore, it is enough to know uniform a priori estimates on the C^0 norm of ψ_t, to ensure the closure of I, and thus the existence of a solution at $t = 1$, i.e. a Kähler-Einstein metric. We recall that by C^0 estimates, we mean, as in most of the literature, a uniform control on $\sup_{x\in X}|\psi_t(x)|$.

In fact, we can even concentrate only on a uniform upper bound on ψ_t (see [Siu88, Proposition 2.1] or [Tia87, pages 235 and 236]).

Notation 6.3. Let us fix some $0 < t_0 \in I$, which exists since $0 \in I$ and I is open.

Let us summarize the consequence of what we have recalled in this section.

Proposition 6.4. *Assume that $[t_0, t_1[\subset I$, that ψ_t denotes the solution at $t \in [t_0, t_1[$, and that there exists a constant C such that $\psi_t \leq C$ $\forall t \in [t_0, t_1[$. Then $t_1 \in I$.*

6.1.2 Reduction to the open orbit

The estimates were obtained by Wang and Zhu [WZ04] in the toric case, by restricting to the open dense torus and using convex analysis. We follow the same general framework, but several modifications are necessary.

Suppose now that X is a $G \times G$-equivariant smooth and Fano compactification of G. Let P be the polytope associated to the anticanonical bundle $-K_X$.

By the action of $K \times K$, if we choose h_{ref} $K \times K$-invariant, we can assume that the functions ψ_t in equation (6.2) are $K \times K$-invariant. A usual way to do this is to consider only the times t for which there exists a $K \times K$-invariant solution and prove openness in this situation, but this is not enough to get an obstruction to the existence of Kähler-Einstein metrics or an upper bound on $R(X)$. To obtain this we use the stronger result that in fact a solution at time t if it exists is unique and thus necessarily $K \times K$-invariant if h_{ref} is. This follows from the uniqueness result for twisted (or generalized) Kähler-Einstein metrics [ZZ14, Corollary 1.4].

By continuity of the solutions ψ_t, it is enough to prove a uniform upper bound on the restrictions of ψ_t to the open and dense orbit $G \subset X$. Let φ_t denote the function on $\mathfrak{a}$ induced by ψ_t. It is enough to give an upper bound for φ_t. We also denote by h_t the hermitian metric on $-K_X$ whose potential with respect to h_{ref} is ψ_t.

Let u_{ref} be the convex potential of h_{ref}, defined on $\mathfrak{a}$, denote by u_t the convex function $u_{\mathrm{ref}} + \varphi_t$ which is the convex potential of the metric h_t. Finally, we denote by w_t the function $tu_t + (1-t)u_{\mathrm{ref}}$.

Proposition 6.5. *Suppose ψ_t is a $K \times K$-invariant solution of equation* (6.2). *Then for $x \in \mathfrak{a}$,*

$$\mathrm{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \alpha(\nabla u_t(x)))^2 = e^{-w_t(x)} J(x). \tag{6.3}$$

Recall that $J(x) = \prod_{\alpha\in\Phi^+} \sinh^2(\alpha(x))$, where Φ is the root system of G.

Proof. We introduced in Section 3.2.3 a left G-invariant section s of the anti-canonical bundle $-K_G$ on G. It gives rise also to a Haar volume form $s^{-1} \wedge \overline{s^{-1}}$ on G. Furthermore, we can express the potential of $(i\partial\overline{\partial}\psi)^n$ with respect to $s^{-1} \wedge \overline{s^{-1}}$, for a smooth function ψ on G, as

$$(i\partial\overline{\partial}\psi)^n = \mathrm{MA}_{\mathbb{C}}(\psi) s^{-1} \wedge \overline{s^{-1}},$$

where $\mathrm{MA}_{\mathbb{C}}(\psi)$ is the complex Monge-Ampère of ψ is the local coordinates given in Section 3.2.3.

Let ψ_{ref} be the potential of the reference metric h_{ref} with respect to the section s and apply this to the function $\psi_{\mathrm{ref}} + \psi_t$. It gives that, on G,

$$\begin{aligned}(\omega_{\mathrm{ref}} + \partial\overline{\partial}\psi_t)|_G^n &= (i\partial\overline{\partial}\psi_{\mathrm{ref}} + \psi_t)^n \\ &= \mathrm{MA}_{\mathbb{C}}(\psi_{\mathrm{ref}} + \psi_t) s^{-1} \wedge \overline{s^{-1}}\end{aligned}$$

Now the computation of the complex Monge-Ampère in local coordinates from Section 3.2.3 gives

$$(\omega_{\mathrm{ref}} + \partial\overline{\partial}\psi_t)^n(\exp(x)) = \mathrm{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha\in\Phi^+} \alpha(\nabla u_t(x)))^2 \frac{1}{J(x)} s^{-1} \wedge \overline{s^{-1}}$$

for $x \in \mathfrak{a}^+$.

On the other hand, the definition of the normalized Ricci potential f_{ref} imply that

$$e^{f_{\mathrm{ref}}} \omega_{\mathrm{ref}}^n = e^{-\psi_{\mathrm{ref}}} s_0^{-1} \wedge \overline{s_0^{-1}},$$

which allows to write the right hand side of equation (6.2) as

$$e^{f_{\mathrm{ref}} - t\psi_t} \omega_{\mathrm{ref}}^n = e^{-t\psi_t - \psi_{\mathrm{ref}}} s_0^{-1} \wedge \overline{s_0^{-1}}.$$

For $x \in \mathfrak{a}$, we have

$$\begin{aligned}-t\psi_t - \psi_{\mathrm{ref}}(\exp(x)) &= -t\varphi_t(x) - u_{\mathrm{ref}}(x) \\ &= -tu_t(x) - (1-t)u_{\mathrm{ref}}(x) \\ &= -w_t(x)\end{aligned}$$

In conclusion, at a point $\exp(x)$ for $x \in \mathfrak{a}^+$, equation (6.2) reads

$$\mathrm{MA}_{\mathbb{R}}(u_t)(x) \prod_{\alpha \in \Phi^+} \alpha(\nabla u_t(x)))^2 \frac{1}{J(x)} s^{-1} \wedge \overline{s^{-1}} = e^{-w_t(x)} s^{-1} \wedge \overline{s^{-1}}.$$

It is equivalent to the equality of the potentials with respect to $s^{-1} \wedge \overline{s^{-1}}$. Furthermore, by multiplying both sides by $J(x)$, we obtain the equation of the statement, that is well defined on the whole of $\mathfrak{a}$, and it is satisfied by W-invariance and smoothness. □

6.1.3 Strategy

To find a uniform upper bound for φ_t we will introduce another function ν_t, and study this function, following the strategy used by Wang and Zhu in the toric case. More precisely, let j be the function on the open Weyl chamber $\mathfrak{a}^+$ defined by $j(x) = -\ln(J(x))$. We consider the function $\nu_t = w_t + j$ defined on $\mathfrak{a}^+$. We will show that it is a strictly convex function on $\mathfrak{a}^+$. It is proper in the following sense: As x goes to infinity, or x goes to a wall of $\mathfrak{a}^+$, $\nu_t(x)$ goes to infinity.

These two properties of ν_t imply that it admits a unique minimum. Let m_t be the minimum of ν_t and x_t be the point of $\mathfrak{a}^+$ where this minimum is attained. We will obtain estimates on both the value m_t of the minimum and on the distance from the origin $|x_t|$ of the point where it is attained.

Then we need to relate these estimates with the function that we want to control. Namely we will go from ν_t to w_t then u_t and finally φ_t.

To summarize, the strategy to prove estimates on φ_t is in three steps:
- reduce to estimates on $|m_t|$ and $|x_t|$,
- find uniform estimates $|m_t| \leq C$,
- get a uniform control $|x_t| \leq C$ of x_t.

We will also have to prove the necessity of the condition and the upper bound on $R(X)$. Before that, we gather some preliminary results.

6.2 Preliminaries

6.2.1 Potentials of metrics in $c_1(X)$

We collect some information about the potentials of smooth hermitian metrics on $-K_X$ with positive curvature that will be used several times in the proof.

The polytope associated to the anticanonical polarization of X is denoted by P, P^+ is its intersection with the positive Weyl chamber and v is the support function of $2P$.

Proposition 6.6. *Let $f : \mathfrak{a} \longrightarrow \mathbb{R}$ be the convex potential of a smooth $K \times K$-invariant hermitian metric on $-K_X$ with positive curvature. Then*

1. *f is W-invariant,*
2. *$\nabla f(\mathfrak{a}) = \mathrm{Int}(2P)$ and $\nabla f(\mathfrak{a}^+) = \mathrm{int}(2P^+)$,*
3. *$|\nabla f| \leq d$ for some constant d independent of f,*
4. *$f(x) \leq v(x - x_0) + f(x_0)$ for any $x_0 \in \mathfrak{a}$, and*
5. *$f(x) \geq v(x) + C_1$ for some constant C_1 depending on f.*

Proof. Since a smooth metric has locally bounded potentials, Proposition 4.26 implies that f is W-invariant and there exists constants C_1 and C_2 depending on f such that

$$v(x) + C_1 \leq f(x) \leq v(x) + C_2.$$

We want to prove the fourth point. Let $x_0 \in \mathfrak{a}$. For any $0 \neq y \in \mathfrak{a}$, consider the slope $\frac{f(x_0+ty)-f(x_0)}{t}$, with $t > 0$. By convexity and the two inequalities given by Proposition 4.26, we see that this slope increases and converges to $v(y)$ as t tends to infinity. This shows that for any $x = x_0 + y \in \mathfrak{a} \setminus \{x_0\}$, we have

$$f(x) \leq v(x - x_0) + f(x_0).$$

This inequality is obviously also satisfied at x_0, so the fourth point is proved.

The second point is exactly the conclusion of Proposition 4.27, and it implies the third, because the polytope P is bounded. Since P contains the origin (by W-invariance), we can take for example d equal the diameter of $2P$. $\square$

Remark 6.7. This proposition in particular applies to the functions u_{ref}, u_t, and w_t.

6.2.2 The functions $j = -\ln(J)$ and ν_t

The aim of this section is to study the functions j and ν_t to show the existence of the minimum of ν_t and obtain some partial estimates on them.

We choose an arbitrary basis $\{e_i\}$ of $\mathfrak{a}$, and corresponding coordinates $\{x_i\}$ when necessary.

Lemma 6.8. *The function j is strictly convex on $\mathfrak{a}^+$.*

Proof. We compute the Hessian of j and check that it is positive definite.

Recall that $J(x) = \prod_{\alpha \in \Phi^+} \sinh^2(\alpha(x))$. Then j is the following sum:

$$j(x) = -2 \sum_{\alpha \in \Phi^+} \ln(\sinh(\alpha(x))).$$

An easy computation shows that

$$\frac{\partial^2}{\partial x_j \partial x_i}(-\ln(\sinh(\alpha(x))) = \alpha(e_i)\alpha(e_j)\frac{1}{\sinh^2(\alpha(x))}.$$

So the Hessian of j is the sum of semipositive matrices, and it is easy to check that the whole sum is definite, so the Hessian of j is positive definite. $\square$

Lemma 6.9. *There exists a constant c such that for any $x \in \mathfrak{a}^+$, we have*

$$j(x) \geq -4\rho(x) + c.$$

Proof. Write

$$\sinh(\alpha(x)) = e^{\alpha(x)}\left(\frac{1 - e^{-2\alpha(x)}}{2}\right) \leq \frac{e^{\alpha(x)}}{2}$$

for $x \in \mathfrak{a}^+$. Then

$$j(x) = -2\sum_{\alpha \in \Phi^+} \ln(\sinh(\alpha(x))) \geq -2\sum_{\alpha \in \Phi^+} \alpha(x) + c,$$

where $c = 2\ln(2)\mathrm{Card}(\Phi^+)$. $\square$

We can now prove the existence of the minimum m_t at x_t.

Lemma 6.10. *The function ν_t admits a unique minimum.*

Proof. We know from Lemma 6.8 that ν_t is strictly convex. We already remarked that $\nu_t(x)$ tends to $+\infty$ when x approaches a Weyl wall. To prove the existence of a minimum x_t it remains to explain why ν_t goes to infinity at infinity.

Proposition 6.6 implies that $w_t(x) \geq v(x) + C_1$ for some constant C_1, where v is the support function of the polytope $2P$, so $\nu_t(x) \geq v(x) + j(x) + C_1$. Then

$$\nu_t(x) \geq v(x) - 4\rho(x) + c + C_1$$

by Lemma 6.9. Finally, the fact that X is Fano implies, by Remark 5.23 that $4\rho \in \mathrm{Int}(P)$, so ν_t is indeed proper. $\square$

The half sum of positive roots ρ is in the interior of $\mathfrak{a}^+$, so $\alpha(\rho) > 0$ for all $\alpha \in \Phi^+$. We will use this as a reference to control the distance to the walls. First, we can say that x_t is not too close to the walls:

Lemma 6.11. *There exists a constant $b_1 > 0$ independent of t such that*

$$x_t \in b_1\rho + \mathfrak{a}^+$$

Proof. By definition of x_t, the derivative of ν_t at x_t vanishes. In particular, the directional derivative of ν_t in the direction ρ is zero:

$$(D\nu_t)_{x_t}(\rho) = \langle \nabla \nu_t(x_t), \rho \rangle =: \frac{\partial \nu_t}{\partial \rho}(x_t) = 0.$$

Recall that $\nu_t = w_t + j$, and that the derivatives of w_t are bounded by Proposition 6.6. In particular we get a bound

$$\left| \frac{\partial w_t}{\partial \rho}(x_t) \right| \leq C.$$

On the other hand, we can compute the directional derivative of j:

$$\frac{\partial j}{\partial \rho}(x_t) = -2 \sum_{\alpha \in \Phi^+} \alpha(\rho) \coth(\alpha(x_t)).$$

So we have

$$\left| 2 \sum_{\alpha \in \Phi^+} \alpha(\rho) \coth(\alpha(x_t)) \right| \leq C$$

but since all the terms of the sum are positive and all the $\alpha(\rho)$ are strictly positive, this implies that for all $\alpha \in \Phi^+$, $\coth(\alpha(x_t)) \leq C$. Observe that the function coth tends to $+\infty$ at 0, so we obtain $\alpha(x_t) \geq C'$ for all α for a constant $C' > 0$ independent of t.

To conclude, observe that the intersection of the half spaces defined by $\alpha(x) \geq C'$ is contained in a translate $b_1 \rho + \mathfrak{a}^+$ for some $b_1 > 0$ sufficiently small, independent of t. □

We will also need to control the derivatives of j away from the walls. This is achieved by the following lemma.

Lemma 6.12. *For any $b > 0$, there exists a constant C such that for any $x \in b\rho + \mathfrak{a}^+$,*

$$|\nabla(j)(x)| \leq C.$$

Proof. Recall that

$$\frac{\partial j}{\partial x_i}(x) = -2 \sum_{\alpha \in \Phi^+} \alpha(e_i) \coth(\alpha(x))$$

For $x \in b\rho + \mathfrak{a}^+$, we have $1 < \coth(\alpha(x)) < \coth(b\alpha(\rho))$, so for any i,

$$\left| \frac{\partial j}{\partial x_i}(x) \right| \leq 2 \sum_{\alpha \in \Phi^+} |\alpha| \coth(b\alpha(\rho)).$$

□

We will also need to control from below the value of ν_t near the walls. This will be achieved by the following technical proposition. For now we cannot control ν_t uniformly close to the walls, but we will as soon as we control m_t. We will use twice the proposition, first to obtain a lower bound on m_t, then to ensure $e^{-\nu_t}$ is sufficiently small near the walls.

Remark also that this proposition can be seen as a precise statement of what we called the properness of ν_t near the walls.

Proposition 6.13. *For any $M > 0$, there exists a constant $b > 0$ independent of t such that for any $x \in \mathfrak{a}^+$ satisfying $\alpha(x) < b\alpha(\rho)$ for some root $\alpha \in \Phi^+$ defining a wall of $\mathfrak{a}^+$, we have*

$$\nu_t(x) \geq m_t + M.$$

Recall that the roots defining the walls are also the simple roots of Φ^+.

Proof. Let $x \in \mathfrak{a}^+$ be such that $\alpha(x) < b_1\alpha(\rho)$ for some simple root $\alpha \in \Phi^+$. Consider the ray $\{x+s\rho, s \geq 0\}$ starting from x. It meets the boundary $\partial(b_1\rho + \mathfrak{a}^+)$ of $b_1\rho + \mathfrak{a}^+$ at a unique point $y = x + s_0\rho$. Furthermore y is in $b_1\rho + \alpha^\perp$ for a simple root α. We can then write $x = y - s_0\rho$, and s_0 satisfies $0 < b_1 - s_0 < b$.

Consider $\alpha \in \Phi^+$ a simple root, and $y \in (b_1\rho + \alpha^\perp) \cap \partial(b_1\rho + \mathfrak{a}^+)$. We will show that there exists a constant $b > 0$ independent of t such that $\nu_t(y - s\rho) \geq m_t + M$ for all s such that $0 < b_1 - s < b$, and that this b can be chosen independent of y and α.

This is enough to prove the proposition because any x as in the statement is of the form above for some α, y and s as shown at the beginning.

Consider the function $g(s) = \nu_t(y - s\rho)$ on $[0, b_1[$. We have $g(0) = \nu_t(y) \geq m_t$ by definition of m_t.

We consider now the derivative of g. Remember that the derivatives of w_t are uniformly bounded, by d, in absolute value by Proposition 6.6. Then

$$g'(s) \geq -d + 2\sum_{\beta \in \Phi^+} \beta(\rho)\coth(\beta(y - s\rho)).$$

Since all the terms in the sum are positive, we have in particular

$$g'(s) \geq -d + 2\alpha(\rho)\coth(\alpha(y - s\rho)).$$

From the assumptions, we compute

$$\alpha(y - s\rho) = b_1\alpha(\rho) - s\alpha(\rho) = (b_1 - s)\alpha(\rho).$$

Observe that the positive function coth is not integrable near 0^+, so

$$\int_0^{b_1} \coth((b_1 - s)\alpha(\rho))ds = +\infty.$$

Together with the fact that g is greater than m_t at 0, it means that for any M, we can find a $b_\alpha > 0$ such that $g(s) \geq M + m_t$ for $b_1 - s \leq b_\alpha$.

Remark that none of what we have done depends on the choice of y. Furthermore, since there are only a finite number of roots α, we can choose a $b > 0$ such that $b < b_\alpha$ for all α, and it concludes the proof. $\square$

6.2.3 Reduction to estimates on m_t and x_t

Lemma 6.14. *Suppose we have uniform estimates $|m_t| < C_m$ and $|x_t| < C_x$ for t in some interval $I \subset [0,1]$. Then there is an uniform upper bound for ϕ_t on I.*

Proof. Recall that it is enough to obtain a uniform upper bound on $u_t - u_{\text{ref}}$ which is a function defined on $\mathfrak{a}$.

We have, by Proposition 6.6 with $x_0 = x_t$, that

$$u_t(x) \leq v(x - x_t) + u_t(x_t)$$

where v is the support function of $2P$. Using the other inequality for u_{ref} we have

$$u_{\text{ref}}(x) \geq v(x) + C_1 \geq v(x - x_t) + C_1 - d|x_t|.$$

Combining these two gives

$$\begin{aligned}(u_t - u_{\text{ref}})(x) &\leq v(x - x_t) + u_t(x_t) - v(x - x_t) - C_1 + d|x_t| \\ &\leq u_t(x_t) - C_1 + d|x_t| \\ &\leq u_t(x_t) - C_1 + dC_x\end{aligned}$$

so we just have to control $u_t(x_t)$.

We have $|m_t| = |\nu_t(x_t)| \leq C_m$, *i.e*

$$|tu_t(x_t) + (1-t)u_{\text{ref}}(x_t) + j(x_t)| \leq C_m.$$

Now we have:

– $t \geq t_0 > 0$,

– $|j(x_t)| \leq C_2$ for some constant C_2 because $x_t \in b_1\rho + \mathfrak{a}^+$,

– and $u_{\text{ref}}(x_t) \leq \sup\{u_{\text{ref}}(y); y \in B(0, C_x)\} =: C_3$.

So

$$u_t(x_t) \leq \frac{C_m + C_2 + C_3}{t_0}.$$

Finally we have proved the uniform upper bound

$$(u_t - u_{\text{ref}})(x) \leq C_4 := \frac{C_m + C_2 + C_3}{t_0} - C_1 + dC_x.$$

□

6.3 Estimates on $|m_t|$

We consider the set

$$A_t := \{x \in \mathfrak{a}^+; m_t \leq \nu_t(x) \leq m_t + 1\} \subset \mathfrak{a}^+.$$

We will obtain upper and lower bound for the volume of A_t. The upper bound will depend on m_t. The key is to obtain an upper bound that is small enough to give information, namely its logarithm has to be strictly dominated by m_t.

In the course of the proof we will use the following property of A_t.

Proposition 6.15. *The set A_t is a bounded and convex set.*

Proof. Since m_t is the minimum of ν_t, A_t is a sublevel set of ν_t which is convex, so A_t is convex. Furthermore, by the properness of ν_t, A_t is a bounded set. $\square$

Lemma 6.16. *There is an upper bound on the volume of A_t:*

$$\mathrm{Vol}(A_t) \leq Ce^{m_t/2}$$

where the constant $C > 0$ does not depend on $t \geq t_0$.

Proof. Fritz John proved in [Joh48, Theorem III] that for any convex and bounded subset A of $\mathbb{R}^r$, there exists an ellipsoid E such that

$$\frac{1}{r}E \subset A \subset E$$

where $\frac{1}{r}E$ is the dilation of E of factor $\frac{1}{r}$ centered at the center of the ellipsoid E.

By Proposition 6.15 we can find such an ellipsoid E_t for A_t. Let T be a linear transformation, of determinant one, such that $T(E) = B(y, \delta)$ is a ball. We will obtain an upper bound on δ, thus getting an upper bound for the volume of $\mathrm{Vol}(A_t)$ because

$$\mathrm{Vol}(A_t) \leq \mathrm{Vol}(E) = \mathrm{Vol}(T(E)) = C\delta^r.$$

Let ν'_t be the function defined by $\nu'_t(x) = \nu_t(T^{-1}(x))$.

We want to use a comparison principle on $B(y, \frac{\delta}{r})$. For that we first show that $\mathrm{MA}_{\mathbb{R}}(\nu'_t)(x) \geq Ce^{-m_t}$ on $T(A_t)$. This is equivalent to showing that $\mathrm{MA}_{\mathbb{R}}(\nu_t)(x) \geq Ce^{-m_t}$ on A_t.

First remark that since the Hessian $\mathrm{Hess}_{\mathbb{R}}\nu_t$ of ν_t satisfies:

$$\mathrm{Hess}_{\mathbb{R}}\nu_t = t\mathrm{Hess}_{\mathbb{R}}u_t + (1-t)\mathrm{Hess}_{\mathbb{R}}u_{\mathrm{ref}} + \mathrm{Hess}_{\mathbb{R}}j,$$

we have

$$\det(\mathrm{Hess}_{\mathbb{R}}\nu_t) \geq \det(t\mathrm{Hess}_{\mathbb{R}}u_t),$$

i.e.

$$\mathrm{MA}_{\mathbb{R}}(\nu_t)(x) \geq t^r \mathrm{MA}_{\mathbb{R}}(u_t)(x).$$

Using Proposition 6.5 we deduce that

$$\begin{aligned}\mathrm{MA}_{\mathbb{R}}(\nu_t)(x) &\geq t^r J(x) e^{-w_t(x)} \prod_{\alpha\in\Phi^+} \frac{1}{\alpha(\nabla u_t(x))^2} \\ &\geq t^r e^{-\nu_t(x)} \prod_{\alpha\in\Phi^+} \frac{1}{\alpha(\nabla u_t(x))^2}.\end{aligned}$$

We treat the factors separately:
- We have $t \geq t_0 > 0$ for t_0 defined in Notation 6.3.
- By definition of A_t, we have $e^{-\nu_t(x)} \geq e^{-m_t-1}$ on A_t.

– For any $x \in \mathfrak{a}$, we have $\nabla u_t(x) \in 2P$, so for any $\alpha \in \Phi^+$, $\alpha(\nabla u_t(x))$ is bounded above independently of t. This implies that

$$\prod_{\alpha \in \Phi^+} \frac{1}{\alpha(\nabla u_t(x))^2} \geq c$$

for some positive constant c.

In conclusion, we indeed have an inequality $\mathrm{MA}_{\mathbb{R}}(\nu_t)(x) \geq Ce^{-m_t}$ on A_t, with C a positive constant independent of $t \geq t_0$.

Now we use the comparison principle on $B(y, \frac{\delta}{r})$ for real Monge-Ampère equations: let g be the auxiliary function defined by

$$g(x) = C^{1/r} e^{-m_t/r}(|x-y|^2 - \frac{\delta^2}{r^2}) + m_t + 1.$$

Then we have

– $g(x) = m_t + 1 \geq \nu'_t(x)$ for $x \in \partial B(y, \frac{\delta}{r})$, and
– $\mathrm{MA}_{\mathbb{R}}(g)(x) = Ce^{-m_t} \leq \mathrm{MA}_{\mathbb{R}}(\nu'_t)(x)$ on $B(y, \frac{\delta}{r})$.

So the comparison principle gives that $\nu'_t(x) \leq g(x)$ on $B(y, \frac{\delta}{r})$. In particular, we have

$$\begin{aligned} m_t &\leq \nu_t(T^{-1}(y)) \\ &\leq \nu'_t(y) \\ &\leq g(y) \\ &\leq C^{1/r} e^{-m_t/r}(-\frac{\delta^2}{r^2}) + m_t + 1. \end{aligned}$$

We deduce from that the following upper bound for δ:

$$\delta \leq \sqrt{\frac{1}{C^{1/r}}} r e^{m_t/2r}.$$

Putting everything together, we obtain

$$\mathrm{Vol}(A_t) \leq \mathrm{Vol}(B(y, \delta) \leq C' e^{m_t/2}.$$

□

We turn now to a lower bound on $\mathrm{Vol}(A_t)$.

Lemma 6.17. *There exists a constant $c > 0$ independent of t such that*

$$\mathrm{Vol}(A_t) \geq c.$$

Proof. There exists a constant b_2 independent of t such that $0 < b_2 < b_1$ and

$$A_t \subset b_2\rho + \mathfrak{a}^+.$$

This is a corollary of Proposition 6.13, taking b_2 corresponding to $M = 1$.

Indeed, by Lemma 6.12 and Proposition 6.6, on $b_2\rho + \mathfrak{a}^+$, $|\nabla(\nu_t)|$ is bounded independently of t, say by M. Then it is clear that the ball $B(x_t, M)$ is contained in A_t. So $\mathrm{Vol}(A_t) \geq c$ for some $c > 0$ independent of t. □

Proposition 6.18. *The following integral is independent of t:*

$$\int_{\mathfrak{a}^+} e^{-\nu_t(x)}dx = V = \int_{2P^+} \prod_{\alpha\in\Phi^+} \alpha(p)^2 dp.$$

Proof. Applying Proposition 4.28 with the ample line bundle $-K_X$, we have, for some constant C depending only on G, and for any convex potential u of a smooth $K\times K$-invariant positively curved hermitian metric on $-K_X$,

$$\begin{aligned}\deg(-K_X) &= C\int_{\mathfrak{a}^+} \prod_{\alpha\in\Phi^+} (\alpha(\nabla u(x)))^2 \mathrm{MA}_{\mathbb{R}}(u)(x)dx \\ &= C\int_{2P^+} \prod_{\alpha\in\Phi^+} \alpha(p)^2 dp\end{aligned}$$

We apply this to the convex potential u_t, which by Proposition 6.5 satisfies

$$e^{-\nu_t(x)} = \prod_{\alpha\in\Phi^+} (\alpha(\nabla u(x)))^2 \mathrm{MA}_{\mathbb{R}}(u)(x)$$

and obtain the statement, with $V = \deg(-K_X)/C$. $\square$

We can now prove the main result of this subsection.

Proposition 6.19.
- *There exists a constant C independent of t, such that $|m_t| \leq C$.*
- *There exist a constant $\kappa > 0$ and a constant C, both independent of t, such that for $x \in \mathfrak{a}^+$,*

$$\nu_t(x) \geq \kappa|x - x_t| - C.$$

Proof. Following here Donaldson [Don08] rather than Wang and Zhu, we write

$$\begin{aligned}\int_{\mathfrak{a}^+} e^{-\nu_t(x)}dx &= \int_{\mathfrak{a}^+}\int_{\nu_t(x)}^{+\infty} e^{-s}dsdx \\ &= \int_{-\infty}^{+\infty} e^{-s} \int_{\mathfrak{a}^+} 1_{\{\nu_t(x)\leq s\}}dxds \\ &= \int_{m_t}^{+\infty} e^{-s}\mathrm{Vol}(\{\nu_t \leq s\})ds \\ &= e^{-m_t}\int_0^{+\infty} e^{-s}\mathrm{Vol}(\{\nu_t \leq m_t + s\})ds\end{aligned}$$

Now remark that $\{\nu_t \leq m_t + s\} \subset s\cdot A_t$ by convexity of ν_t, where $s\cdot A_t$ is the s-dilation of A_t with center x_t. We deduce from that

$$\mathrm{Vol}(\{w_t \leq \nu_t + s\}) \leq s^r \mathrm{Vol}(A_t) \leq Cs^r e^{m_t/2}.$$

Applying this to the formula above we obtain

$$\begin{aligned}\int_{\mathfrak{a}^+} e^{-\nu_t(x)}dx &\leq e^{-m_t}Ce^{m_t/2}\int_0^{+\infty} e^{-s}s^n ds\\ &\leq C'e^{-m_t/2}.\end{aligned}$$

The left hand side being constant, this inequality gives an upper bound on m_t.

For the lower bound, remark that

$$\begin{aligned}V = \int_{\mathfrak{a}^+} e^{-\nu_t(x)}dx = e^{-m_t}&\int_0^{+\infty} e^{-s}\mathrm{Vol}(\{w_t \leq \nu_t + s\})ds\\ \geq e^{-m_t}&\int_1^{+\infty} e^{-s}\mathrm{Vol}(\{w_t \leq \nu_t + s\})ds\\ \geq e^{-m_t}&\mathrm{Vol}(A_t)\int_1^{+\infty} e^{-s}ds.\end{aligned}$$

By Lemma 6.17, $\mathrm{Vol}(A_t)$ admits a lower bound c independent of t, so

$$V \geq e^{-m_t}c\int_1^{+\infty} e^{-s}ds,$$

and we deduce that

$$-m_t \leq \ln(V) - \ln\left(c\int_1^{+\infty} e^{-s}ds\right)$$

so $-m_t$ is bounded above by a constant independent of t. Thus we have showed estimates on $|m_t|$.

Now for linear growth, the estimate on $|m_t|$ implies that we know both an upper bound C_1 and a lower bound C_2 independent of t for the volume of A_t. Since this set is convex, and contains a ball $B(x_t, \delta_0)$ of fixed radius δ_0 independent of t by the proof of Lemma 6.17, this implies that A_t is included in a ball $A_t \subset B(x_t, \delta)$ where δ only depends on C_1 and δ_0.

By convexity of ν_t, this implies that $\nu_t(x) \geq \frac{1}{\delta}|x - x_t| + m_t$ outside of the ball, and we can extend this inequality to the whole of $\mathfrak{a}^+$ simply by subtracting 1:

$$\nu_t(x) \geq \frac{1}{\delta}|x - x_t| + m_t - 1$$

everywhere. Using again the fact that m_t is uniformly bounded we get the result. $\square$

6.4 Obstruction, and upper bound on $R(X)$

Everything relies on the following vanishing statement.

Proposition 6.20. *Let u be the convex potential of a smooth positive metric on $-K_X$. Define $\nu = u + j$. Let ξ be any vector in $\overline{\mathfrak{a}^+}$. Then*

$$\int_{\mathfrak{a}^+} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = 0.$$

Before we get to the proof, let us remark that the function considered is integrable. More generally, we can remark first that for any potential u_0, and any vector ξ, the function $\frac{\partial u_0}{\partial \xi} e^{-\nu}$ is integrable on $\mathfrak{a}^+$. This is the case because $\nabla u_0 \in 2P$, and $e^{-\nu} \leq C e^{-(v-4\rho)+C}$ (by Proposition 6.6 and Lemma 6.9) is obviously integrable.

Secondly, we have to show that the function $\frac{\partial j}{\partial \xi} e^{-\nu} = \frac{\partial j}{\partial \xi} J e^{-u}$ is integrable. Write

$$\begin{aligned}
\frac{\partial j}{\partial \xi}(x) J(x) &= -2 \sum_{\alpha \in \Phi^+} \alpha(\xi) \mathrm{coth}(\alpha(x)) \prod_{\beta \in \Phi^+} \sinh^2(\beta(x)) \\
&= -2 \sum_{\alpha \in \Phi^+} \alpha(\xi) \mathrm{cosh}(\alpha(x)) \mathrm{sinh}(\alpha(x)) \prod_{\beta \neq \alpha} \sinh^2(\beta(x))
\end{aligned}$$

Then by a computation similar to Lemma 6.9, we have

$$|e^{4\rho} \frac{\partial j}{\partial \xi}(x) J(x)| \leq C e^{4\rho},$$

so again $\frac{\partial j}{\partial \xi} e^{-\nu}$ is integrable.

Proof. Choose a basis $(e_i)_{i=1..s}$ of the semisimple part $\mathfrak{a}_{ss}$ which generate the Weyl chamber as a cone, and a basis $(f_j)_{j=1..r-s}$ of the central part $\mathfrak{a}_z$. Consider the sets

$$Q(\epsilon, M) := \left\{ \sum_i x_i e_i + \sum_j y_j f_j ; \forall i \ \epsilon \leq x_i \leq M, \forall j \ -M \leq y_j \leq M \right\}$$

for $0 \leq \epsilon < M$. Let $S_1(\epsilon, M) = \{\sum_i x_i e_i + \sum_j y_j f_j \in Q(\epsilon, M); \exists i \ x_i = \epsilon\}$ and $S_2(\epsilon, M) = \{\sum_i x_i e_i + \sum_j y_j f_j \in Q(\epsilon, M); \exists i \ x_i = M \text{ or } \exists j \ |y_j| = M\}$. Remark that $\partial Q(\epsilon, M) = S_1(\epsilon, M) \cup S_2(\epsilon, M)$.

Remark that $\frac{\partial \nu}{\partial \xi} e^{-\nu} = -\frac{\partial e^{-\nu}}{\partial \xi}$. Then by the divergence formula applied to $e^{-\nu}$ we have for $\epsilon > 0$,

$$\int_{Q(\epsilon, M)} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = \int_{S_1(\epsilon, M) \cup S_2(\epsilon, M)} e^{-\nu} \langle \xi, \mu \rangle \, d\sigma$$

where μ is the exterior normal and $d\sigma$ is the surface area.

Write now $e^{-\nu} = e^{-u} J$. This is a continuous function on $\mathfrak{a}$, and it vanishes on the Weyl walls. Fixing M, we can thus let ϵ tend to 0, and we have that $e^{-\nu}$ tends uniformly to 0 on $S_1(\epsilon, M)$, so $\int_{S_1(\epsilon, M)} e^{-\nu} \langle \xi, \mu \rangle \, d\sigma$ tends to 0.

We thus have

$$\int_{Q(0,M)} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = \int_{S_2(0,M)} e^{-\nu} \langle \xi, \mu \rangle \, d\sigma.$$

Now as we have seen before, we have $e^{-\nu} \leq Ce^{-(v-4\rho)+C}$, so $e^{-\nu}(x)$ decreases exponentially as $|x|$ tends to infinity. Since the area of $S_2(0,M)$ grows polynomially, this ensures that $\int_{S_2(0,M)} e^{-\nu} \langle \xi, \mu \rangle \, d\sigma$ tends to zero as M tends to ∞. This ends the proof. □

Let us apply this to prove our obstruction to the existence of a Kähler-Einstein metric.

Proposition 6.21. *Assume there exists a Kähler-Einstein metric on X. Then*

$$\mathrm{bar}_{DH}(2P^+) \in 4\rho + \Xi.$$

Proof. We have seen that the Kähler-Einstein equation restricted to the open orbit reads:

$$\mathrm{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \alpha(\nabla u)^2 = e^{-u} J.$$

Suppose that there exists a solution u.

Applying Proposition 6.20 to u gives

$$\int_{\mathfrak{a}^+} \frac{\partial \nu}{\partial \xi} e^{-\nu} dx = 0,$$

so by definition of $\nu = u + j$,

$$\int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} e^{-\nu} dx = -\int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu} dx.$$

Since u is solution to the Kähler-Einstein equation, we have

$$e^{-\nu} = e^{-u} J = \mathrm{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \alpha(\nabla u)^2.$$

In particular,

$$\int_{\mathfrak{a}^+} e^{-\nu} dx = V$$

is constant.

Since we have the inequality $\frac{\partial j}{\partial \xi} \leq -4\rho(\xi)$ everywhere for $\xi \in \overline{\mathfrak{a}^+}$, we have

$$-\int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi} e^{-\nu} dx \geq 4\rho(\xi) V.$$

So we have

$$\begin{aligned} 4\rho(\xi) V &\leq \int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} e^{-\nu} dx \\ &\leq \int_{\mathfrak{a}^+} \frac{\partial u}{\partial \xi} \mathrm{MA}_{\mathbb{R}}(u) \prod_{\alpha \in \Phi^+} \alpha(\nabla u)^2 dx. \end{aligned}$$

Using the Legendre transform, we can rewrite the left-hand side as

$$\begin{aligned}&\leq \int_{2P^+} \langle p, \xi\rangle \prod_{\alpha\in\Phi^+} \langle \alpha, p\rangle^2 \, dp \\ &\leq \left\langle \xi, \mathrm{bar}_{DH}(2P^+)\right\rangle V.\end{aligned}$$

Dividing by V we obtain that for any $\xi \in \mathfrak{a}^+$,

$$\left\langle \xi, \mathrm{bar}_{DH}(2P^+)\right\rangle \geq 4\rho(\xi)$$

and that precisely means

$$\mathrm{bar}_{DH}(2P^+) - 4\rho \in (\mathfrak{a}^+)^\vee.$$

We have also that $(\mathfrak{a}^+)^\vee \subset \mathfrak{a}_{ss}$, and the inequality $\frac{\partial j}{\partial \xi} \leq -4\rho(\xi)$ is in fact strict when $\xi \in \mathfrak{a}^+_{ss}$, so indeed

$$\mathrm{bar}_{DH}(2P^+) - 4\rho \in \Xi.$$

□

We now turn to an upper bound on $R(X)$.

Proposition 6.22. *Assume that X admits no Kähler-Einstein metrics, then the greatest Ricci lower bound of X is lower than or equal to the supremum of all $t < 1$ such that*

$$4\rho + \frac{t}{t-1}(\mathrm{bar}_{DH}(2P^+) - 4\rho) \in 2P^+ + (-\Xi).$$

Proof. Consider the equation at time t:

$$\mathrm{MA}_{\mathbb{R}}(u_t) \prod_{\alpha\in\Phi^+} \alpha(\nabla u_t)^2 = e^{-\nu_t}.$$

Apply the proposition to w_t. This gives for any $\xi \in \overline{\mathfrak{a}^+}$,

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi} e^{-\nu_t} dx = 0.$$

This is equivalent to

$$t \int \frac{\partial u_t}{\partial \xi} e^{-\nu_t} + (1-t) \int \frac{\partial u_{\mathrm{ref}}}{\partial \xi} e^{-\nu_t} + \int \frac{\partial j}{\partial \xi} e^{-\nu_t} = 0.$$

Without loss of generality we can assume $t < 1$ and divide by $(t-1)V$ to get

$$\frac{t}{t-1} \int \frac{\partial u_t}{\partial \xi} \frac{e^{-\nu_t}}{V} + \frac{1}{t-1} \int \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} = \int \frac{\partial u_{\mathrm{ref}}}{\partial \xi} \frac{e^{-\nu_t}}{V}.$$

If v is the support function of $2P$, we have for any ξ and at any $x \in \mathfrak{a}^+$, $\frac{\partial u_{\mathrm{ref}}}{\partial \xi}(x) \leq v(\xi)$, so

$$\int \frac{\partial u_{\mathrm{ref}}}{\partial \xi} \frac{e^{-\nu_t}}{V} \leq v(\xi).$$

On the other hand, we can use here also the fact that $\frac{\partial j}{\partial \xi} \leq -4\rho(\xi)$ for $\xi \in \overline{\mathfrak{a}^+}$ to get

$$\frac{1}{t-1} \int \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} \geq \frac{1}{t-1}(-4\rho(\xi)).$$

We thus have

$$\begin{aligned} v(\xi) &\geq \frac{t}{t-1} \int \frac{\partial u_t}{\partial \xi} \frac{e^{-\nu_t}}{V} + \frac{1}{t-1} \int \frac{\partial j}{\partial \xi} \frac{e^{-\nu_t}}{V} \\ &\geq \frac{t}{t-1} \left\langle \xi, \mathrm{bar}_{DH}(2P^+) \right\rangle - \frac{1}{t-1} 4\rho(\xi) \\ &\geq \left\langle \xi, 4\rho + \frac{t}{t-1}(\mathrm{bar}_{DH}(2P^+) - 4\rho) \right\rangle \end{aligned}$$

The fact that this is true $\forall \xi \in \overline{\mathfrak{a}^+}$ means, since v is the support function of $2P$,

$$4\rho + \frac{t}{t-1}(\mathrm{bar}_{DH}(2P^+) - 4\rho) \in 2P^+ + (-\Xi).$$

□

6.5 Absence of estimates on $|x_t|$

We will assume now that there are no Kähler-Einstein metrics on X. We will denote by $t_\infty := R(X)$ the greatest Ricci lower bound.

Our assumption implies that $|x_t|$ is unbounded as t tends to t_∞. Indeed if it was not the case, then we would have estimates on $|x_t|$ and so by all what we have done here, there would be a solution at time t_∞ and by openness for times greater than t_∞. This is a contradiction.

We can find a sequence t_i such that $t_i \to t_\infty$ and $|x_{t_i}| \to \infty$. Let $\xi_t = \frac{x_t}{|x_t|} \in \mathfrak{a}^+$. Up to taking a subsequence, we can also assume that ξ_t admits a limit $\xi_\infty \in \overline{\mathfrak{a}^+}$ as $t_i \to t_\infty$.

We will consider an integral equality involving ν_t and consider the limit as $t_i \to t_\infty$. The integral equality follows from the vanishing result already proved (Proposition 6.20) applied to w_t :

Lemma 6.23. *We have*

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi} e^{-\nu_t} dx = 0$$

Recall that $w_t = tu_t + (1-t)u_{\mathrm{ref}}$ by definition, so

$$\nu_t = tu_t + (1-t)u_{\mathrm{ref}} + j = t(u_t + j) + (1-t)(u_{\mathrm{ref}} + j).$$

So the vanishing integral of Lemma 6.23 gives

$$t\int_{\mathfrak{a}^+}\frac{\partial u_t+j}{\partial \xi_t}e^{-\nu_t}dx+(1-t)\int_{\mathfrak{a}^+}\frac{\partial u_{\mathrm{ref}}+j}{\partial \xi_t}e^{-\nu_t}dx=0,$$

which can also be written

$$t\left(\int_{\mathfrak{a}^+}\frac{\partial u_t}{\partial \xi_t}e^{-\nu_t}dx+\int_{\mathfrak{a}^+}\frac{\partial j}{\partial \xi_t}e^{-\nu_t}dx\right)=(t-1)\int_{\mathfrak{a}^+}\frac{\partial u_{\mathrm{ref}}+j}{\partial \xi_t}e^{-\nu_t}dx.$$

We will compute the limit of each of these terms as $t_i\to t_\infty$.

Let us first consider $\int_{\mathfrak{a}^+}\frac{\partial u_t}{\partial \xi_t}e^{-\nu_t}dx$. Let $\mathrm{bar}_{DH}(2P^+)$ denote the barycenter of $2P^+$ with respect to the measure $\prod_{\alpha\in\Phi^+}(\alpha(p))^2dp$. Recall also that $V=\int_{\mathfrak{a}^+}e^{-\nu_t}$ is constant.

Lemma 6.24. *We have*

$$\int_{\mathfrak{a}^+}\frac{\partial u_t}{\partial \xi_t}e^{-\nu_t}dx=\big\langle \xi_t,\mathrm{bar}_{DH}(2P^+)\big\rangle V.$$

Proof. Recall that

$$e^{-\nu_t}=\prod_{\alpha\in\Phi^+}(\alpha(\nabla(u_t)))^2\mathrm{MA}_{\mathbb{R}}(u_t).$$

Using the change of variables given by $\nabla(u_t)$, we get

$$\int_{\mathfrak{a}^+}\frac{\partial u_t}{\partial \xi_t}e^{-\nu_t}dx=\int_{2P^+}\langle \xi_t,p\rangle\prod_{\alpha\in\Phi^+}(\alpha(p))^2dp.$$

The result follows by observing that $V=\int_{2P^+}\prod_{\alpha\in\Phi^+}(\alpha(p))^2dp$ by the same change of variables, and that

$$\int_{2P^+}p\prod_{\alpha\in\Phi^+}(\alpha(p))^2dp=\mathrm{bar}_{DH}(2P^+)V.$$

∎

In particular, the limit as $t_i\to t_\infty$ is

$$\big\langle \xi_\infty,\mathrm{bar}_{DH}(2P^+)\big\rangle V.$$

For the other terms we need more work to compute the limits. We will prove the two following propositions.

Proposition 6.25. *We have*

$$\lim_{t_i\to t_\infty}\int_{\mathfrak{a}^+}\frac{\partial j}{\partial \xi_t}e^{-\nu_t}=-4\rho(\xi_\infty)V.$$

Proposition 6.26. *We have*

$$\lim_{t_i \to t_\infty} \int_{\mathfrak{a}^+} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t} e^{-\nu_t} = v(\xi_\infty) V.$$

We will first find a domain $D(\epsilon)$ of the form $B(x_t, \delta) \cap (b\rho + \mathfrak{a}^+)$ where $e^{-\nu_t}dx$ puts all the mass up to $2\epsilon > 0$. When we write $B(x_t, \delta)$, we in general mean $B(x_t, \delta) \cap \mathfrak{a}^+$.

Lemma 6.27. *For any $\epsilon > 0$, there exists a constant $\delta = \delta(\epsilon)$ independent of t such that*

$$\int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\nu_t} dx < \epsilon$$

and

$$e^{-\kappa\delta + C} \sigma_r \delta^{r-1} < \epsilon,$$

where σ_r is the area of a sphere of radius 1 in $\mathbb{R}^r$.

Proof. Recall from Proposition 6.19 that $\nu_t(x) \geq \kappa|x - x_t| - C$, for some $\kappa > 0, C$ independent of t. Observe that the function $e^{-\kappa|x-x_t|+C}$ is well defined on $\mathfrak{a}$, positive and integrable. So there exists a $\delta > 0$ such that

$$\int_{\mathfrak{a} \setminus B(x_t, \delta)} e^{-\kappa|x - x_t| + C} dx < \epsilon.$$

But then we also have

$$\int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\nu_t} dx \leq \int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} e^{-\kappa|x - x_t| + C} < \epsilon.$$

Of course, since $e^{-\kappa y + C}$ decreases exponentially with respect to y, we can increase δ so as to have the second condition:

$$e^{-\kappa\delta + C} \sigma_r \delta^{r-1} < \epsilon.$$

□

Lemma 6.28. *For any $\epsilon > 0$, let $\delta = \delta(\epsilon)$ be given by Lemma 6.27. There exists a constant $b = b(\epsilon) > 0$ such that if we denote by $D = D(\epsilon)$ the domain $B(x_t, \delta) \cap (b\rho + \mathfrak{a}^+)$, we have*

$$\int_{B(x_t, \delta) \setminus D} e^{-\nu_t} dx < \epsilon,$$

and

$$\int_{\partial D} e^{-\nu_t} d\sigma < \epsilon,$$

where $d\sigma$ is the area measure of ∂D, which is piecewise smooth.

Proof. Here we want to use the Proposition 6.13. Now that we know that m_t is uniformly bounded, we can choose M and the corresponding b so that:

$$\forall x \in \mathfrak{a}^+ \setminus (b\rho + \mathfrak{a}^+),\ e^{-\nu_t(x)} < \max\left(\frac{\epsilon}{\sigma_r \delta^{r-1}}, \frac{\epsilon}{\delta^r \omega_r}\right)$$

where ω_r is the volume of the ball of radius 1 in $\mathbb{R}^r$.

Let us prove that $e^{-\nu_t} < \frac{\epsilon}{\sigma_r \delta^{r-1}}$ on ∂D. A point $x \in \partial D$ is either on the sphere of radius δ centered at x_t, or on $\partial(b\rho + \mathfrak{a}^+)$. In the first case, we have, by Proposition 6.19,

$$\begin{aligned} e^{-\nu_t(x)} &\leq e^{-\kappa|x - x_t| + C} \\ &\leq e^{-\kappa\delta + C} \\ &< \frac{\epsilon}{\sigma_r \delta^{r-1}} \end{aligned}$$

by the second consequence of Lemma 6.27. In the second case, $x \in \partial(b\rho + \mathfrak{a}^+)$, so by the choice of b above, using the first term in the maximum, we have

$$e^{-\nu_t(x)} < \frac{\epsilon}{\sigma_r \delta^{r-1}}.$$

Obviously the volume of ∂D is $\leq \sigma_r \delta^{r-1}$, so

$$\begin{aligned} \int_{\partial D} e^{-\nu_t} d\sigma &< \int_{\partial D} \frac{\epsilon}{\sigma_r \delta^{r-1}} d\sigma \\ &< \epsilon. \end{aligned}$$

For the other part, we use the fact that $e^{-\nu_t(x)} < \frac{\epsilon}{\delta^r \omega_r}$ on $B(x_t, \delta) \setminus D \subset \mathfrak{a}^+ \setminus (b\rho + \mathfrak{a}^+)$, which implies that

$$\int_{B(x_t,\delta)\setminus D} e^{-\nu_t} dx < \int_{B(x_t,\delta)\setminus D} \frac{\epsilon}{\delta^r \omega_r} dx < \epsilon$$

using the fact that the volume of $B(x_t, \delta) \setminus D$ is $\leq \delta^r \omega_r$. □

Lemma 6.29. *Let $\epsilon > 0$ and $D = D(\epsilon)$ be the domain given by Lemma 6.28. We have*

$$\left| \int_{\mathfrak{a}^+ \setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} \right| < (2d + 1)\epsilon.$$

Proof. Let us write:

$$\begin{aligned} \int_{\mathfrak{a}^+\setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} dx &= \int_{\mathfrak{a}^+\setminus D} \frac{\partial(\nu_t - w_t)}{\partial \xi_t} e^{-\nu_t} dx \\ &= \int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx - \int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx - \int_{\mathfrak{a}^+\setminus D} \frac{\partial w_t}{\partial \xi_t} e^{-\nu_t} dx \end{aligned}$$

The first of these three integrals is zero by Lemma 6.23.

For the third term, we have

$$\int_{\mathfrak{a}^+\setminus D} \left|\frac{\partial w_t}{\partial \xi_t}\right| e^{-\nu_t} dx \leq d \int_{\mathfrak{a}^+\setminus D} e^{-\nu_t} dx$$
$$< 2d\epsilon$$

by Lemma 6.28.

It remains to deal with the second integral. We apply the divergence theorem on D to the function $e^{-\nu_t}$:

$$\int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx = -\int_D \frac{\partial e^{-\nu_t}}{\partial \xi_t} dx = \int_{\partial D} e^{-\nu_t} \langle \xi_t, n\rangle \, d\sigma$$

where n is the exterior normal to D and $d\sigma$ the area measure.

We conclude using Lemma 6.28 that

$$\left|\int_D \frac{\partial \nu_t}{\partial \xi_t} e^{-\nu_t} dx\right| \leq \int_{\partial D} e^{-\nu_t} d\sigma < \epsilon.$$

Putting everything together, we see that

$$\left|\int_{\mathfrak{a}^+\setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} dx\right| < (2d+1)\epsilon$$

□

Proof of Proposition 6.25. Let $\epsilon > 0$. Set $\theta = \frac{\epsilon}{3(2d+1+8|\rho|)}$ and let $D := D(\theta)$. Write

$$\left|\int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_\infty)V\right| \leq \left|\int_{\mathfrak{a}^+\setminus D} \frac{\partial j}{\partial \xi_t} e^{-\nu_t}\right| + \left|\int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_\infty)V\right|$$
$$\leq (2d+1)\theta + \left|\int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_\infty)V\right|$$

by Lemma 6.29.

Then we have

$$\left|\int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_\infty)V\right| \leq \left|\int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_t)V\right| + |4\rho(\xi_\infty - \xi_t)V|$$

The second term tends to zero so there exists an i_0 such that for all $i \geq i_0$,

$$|4\rho(\xi_\infty - \xi_t)V| \leq \frac{\epsilon}{3}.$$

We now deal with the second term:

$$\begin{aligned}\left|\int_D \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_t)V\right| &\leq \left|\int_D \left(\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right) e^{-\nu_t}\right| + \left|\int_{\mathfrak{a}^+\setminus D} 4\rho(\xi_t) e^{-\nu_t}\right| \\ &\leq \left|\int_D \left(\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right) e^{-\nu_t}\right| + \left|4\rho(\xi_t)\int_{\mathfrak{a}^+\setminus D} e^{-\nu_t}\right| \\ &\leq \left|\int_D \left(\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right) e^{-\nu_t}\right| + |4\rho|\left|\int_{\mathfrak{a}^+\setminus D} e^{-\nu_t}\right|\end{aligned}$$

and by construction of D we deduce:

$$\leq \left|\int_D \left(\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right) e^{-\nu_t}\right| + |4\rho| \cdot 2\theta$$

We consider now the quantity

$$\frac{\partial j}{\partial \xi_t}(x) + 4\rho(\xi_t)$$

for $x \in D$.

The first thing to remark is that it is negative. Indeed, recall that

$$\begin{aligned}\frac{\partial j}{\partial \xi_t}(x) &= -2\sum_{\alpha\in\Phi^+} \alpha(\xi_t)\coth(\alpha(x)) \\ &\leq -2\sum_{\alpha\in\Phi^+} \alpha(\xi_t) \\ &= -4\rho(\xi_t)\end{aligned}$$

Recall that $D(\theta) \subset b(\theta)\rho + \mathfrak{a}^+$ for some $b(\theta) > 0$, and more precisely that $D(\theta) = B(x_t, \delta(\theta)) \cap (b(\theta)\rho + \mathfrak{a}^+)$. Choose $b_0 > 0$ such that $B(b_0\rho, \delta(\theta)) \subset b(\theta)\rho + \mathfrak{a}^+$.

We can write

$$\begin{aligned}\xi_t &= \frac{x_t}{|x_t|} \\ &= \frac{x_t - b_0\rho}{|x_t|} + \frac{b_0\rho}{|x_t|} \\ &= \frac{|x_t - b_0\rho|}{|x_t|}\xi_1 + \frac{|b_0\rho|}{|x_t|}\xi_2\end{aligned}$$

where $\xi_1 = \frac{x_t - b_0\rho}{|x_t - b_0\rho|}$ and $\xi_2 = \frac{b_0\rho}{|b_0\rho|}$.

It gives

$$\frac{\partial}{\partial \xi_t} = \frac{|x_t - b_0\rho|}{|x_t|}\frac{\partial}{\partial \xi_1} + \frac{|b_0\rho|}{|x_t|}\frac{\partial}{\partial \xi_2}.$$

Let $x = x_t + y \in D$, we consider the restriction of j to the line starting from $b_0\rho + y$ and of direction ξ_1, which contains x. By convexity, we have

$$\frac{\partial j}{\partial \xi_1}(x) \geq \frac{j(x) - j(y + b_0\rho)}{|x_t - b_0\rho|}.$$

Recall from Lemma 6.9 that

$$j(x) \geq -2 \sum_{\alpha\in\Phi^+} \alpha(x) + C = -4\rho(x) + C$$

on $\mathfrak{a}^+$, for some constant C.

Applying this gives

$$\frac{\partial j}{\partial \xi_1}(x) \geq \frac{-4\rho(x) + C - j(y + b_0\rho)}{|x_t - b_0\rho|}.$$

Now going back to $\frac{\partial j}{\partial \xi_t}(x)$, we have

$$\begin{aligned}
\frac{\partial j}{\partial \xi_t}(x) &= \frac{|x_t - b_0\rho|}{|x_t|}\frac{\partial j}{\partial \xi_1}(x) + \frac{|b_0\rho|}{|x_t|}\frac{\partial j}{\partial \xi_2}(x) \\
&\geq \frac{|x_t - b_0\rho|}{|x_t|}\frac{-4\rho(x) + C - j(y + b_0\rho)}{|x_t - b_0\rho|} + \frac{|b_0\rho|}{|x_t|}\frac{\partial j}{\partial \xi_2}(x) \\
&\geq \frac{-4\rho(x_t + y) + C - j(y + b_0\rho)}{|x_t|} + \frac{|b_0\rho|}{|x_t|}\frac{\partial j}{\partial \xi_2}(x)
\end{aligned}$$

so

$$0 \geq \frac{\partial j}{\partial \xi_t}(x) + 4\rho\left(\frac{x_t}{|x_t|}\right) \geq \frac{-4\rho(y) + C - j(y + b_0\rho) + |b_0\rho|\frac{\partial j}{\partial \xi_2}(x)}{|x_t|}$$

Now $y \in B(b_0\rho, \delta(\theta))$ is bounded, j is bounded on $b(\theta)\rho + \mathfrak{a}^+$, and the derivatives of j are bounded on $b(\theta)\rho + \mathfrak{a}^+$, so there is a (negative) constant C' such that

$$0 \geq \frac{\partial j}{\partial \xi_t}(x) + 4\rho\left(\frac{x_t}{|x_t|}\right) \geq \frac{C'}{|x_t|}.$$

Applying this to the sequence t_i, we find that there exists a i_1 such that for $i > i_1$, and uniformly for $x \in D$,

$$\left|\frac{\partial j}{\partial \xi_t}(x) + 4\rho\left(\xi_t\right)\right| \leq \frac{\epsilon}{3V}.$$

Then for $i > i_1$,

$$\begin{aligned}
\left|\int_D \left(\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right) e^{-\nu_t}\right| &\leq \int_D \left|\frac{\partial j}{\partial \xi_t} + 4\rho(\xi_t)\right| e^{-\nu_t} \\
&\leq \frac{\epsilon}{3V}\int_D e^{-\nu_t} \\
&\leq \frac{\epsilon}{3}
\end{aligned}$$

Gathering everything gives, for $i > \max(i_0, i_1)$,

$$\left| \int_{\mathfrak{a}^+} \frac{\partial j}{\partial \xi_t} e^{-\nu_t} + 4\rho(\xi_\infty) V \right| \leq (2d+1)\theta + 8|\rho|\theta + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.$$

□

Proof of Proposition 6.26. Let $\epsilon > 0$. Set $\theta := \frac{\epsilon}{6d}$ and let $\delta = \delta(\theta)$. First, by Lemma 6.27, we have

$$\left| \int_{\mathfrak{a}^+ \setminus B(x_t, \delta)} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t} e^{-\nu_t} \right| < d\theta.$$

On $B(x_t, \delta)$, we always have $\frac{\partial u_{\mathrm{ref}}}{\partial \xi_t} \leq v(\xi_t)$. Now consider the ray starting from $x - x_t$ and going to x. By convexity, we have

$$\begin{aligned} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t}(x) &\geq \frac{u_{\mathrm{ref}}(x) - u_{\mathrm{ref}}(x - x_t)}{|x_t|} \\ &\geq \frac{v(x) + C}{|x_t|} \end{aligned}$$

for some constant C independent of x in $B(x_t, \delta)$, by Proposition 6.6 and because u_{ref} is bounded on $B(0, \delta)$. Then we can write

$$\begin{aligned} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t}(x) &\geq v(\xi_t + \frac{x - x_t}{|x_t|}) + \frac{C}{|x_t|} \\ &\geq v(\xi_t) + \frac{C'}{|x_t|} \end{aligned}$$

The last step holds because v is Lipschitz.

For $i > i_0$ for some i_0, we thus have, for $x \in B(x_t, \delta)$,

$$\left| \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t}(x) - v(\xi_t) \right| < \frac{\epsilon}{3} V.$$

Integrating on the ball gives

$$\left| \int_{B(x_t, \delta)} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t} e^{-\nu_t} - \int_{B(x_t, \delta)} v(\xi_t) e^{-\nu_t} \right| < \frac{\epsilon}{3}.$$

Applying Lemma 6.27 again gives

$$\left| \int_{B(x_t, \delta)} v(\xi_t) e^{-\nu_t} - \int_{\mathfrak{a}^+} v(\xi_t) e^{-\nu_t} \right| < d\theta,$$

with

$$\int_{\mathfrak{a}^+} v(\xi_t) e^{-\nu_t} = v(\xi_t) V.$$

Finally, since ξ_t converges to ξ_∞, there exists i_1 such that for $i > i_1$,

$$|v(\xi_t)V - v(\xi_\infty)V| < \frac{\epsilon}{3}.$$

We have proved that for $i > i_0, i_1$, we have

$$\left| \int_{\mathfrak{a}^+} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_t} e^{-\nu_t} - v(\xi_\infty)V \right| < 2\frac{\epsilon}{3} + 2d\theta = \epsilon$$

□

6.5.1 Conclusion

We can now prove that our condition is sufficient for the existence of a Kähler-Einstein metric.

Theorem 6.30. *If* $\mathrm{bar}_{DH}(2P^+) \in 4\rho + \Xi$*, then* X *admits a Kähler-Einstein metric.*

Proof. Assume first that X admits no Kähler-Einstein metric. Then as $t_i \to t_\infty$ we have the equality

$$t_\infty(\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty)V = (t_\infty - 1)(v - 4\rho)(\xi_\infty)V$$

or, dividing by V,

$$t_\infty(\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty) = (t_\infty - 1)(v - 4\rho)(\xi_\infty).$$

In particular, since v is the support function of $2P$ and $2\rho \in \mathrm{Int}(P)$, and $t_\infty \leq 1$, we have

$$t_\infty(\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty) \leq 0.$$

Assume that $\mathrm{bar}_{DH}(2P^+) \in 4\rho + \Xi$. Then by the definition of Ξ, the only possibility is that $\xi_\infty \in \mathfrak{a}_t$ and $(\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty) = 0$.

To prove that this is impossible we have to give a slightly different proof. It is simpler and in fact the same as in the toric case. We consider the vanishing

$$\int_{\mathfrak{a}^+} \frac{\partial \nu_t}{\partial \xi_\infty} e^{-\nu_t} dx = 0.$$

The difference with what we have done before is that we fix ξ_∞ instead of considering ξ_t.

Since $\xi_\infty \in \mathfrak{a}_t$, we have $\frac{\partial j}{\partial \xi_\infty} = 0$ and so we deduce from the vanishing of the integral the following equality, valid for $t < t_\infty$.

$$t \int_{\mathfrak{a}^+} \frac{\partial u_t}{\partial \xi_\infty} e^{-\nu_t} dx = (t-1) \int_{\mathfrak{a}^+} \frac{\partial u_{\mathrm{ref}}}{\partial \xi_\infty} e^{-\nu_t} dx$$

The left hand side term is zero because we assumed

$$0 = (\mathrm{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty) = \mathrm{bar}_{DH}(2P^+)(\xi_\infty).$$

We thus have, for all $t < t_\infty$,

$$\int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_\infty} e^{-\nu_t} dx = 0.$$

This is a contradiction: let $m := \min\{v(\xi); \xi \in \mathfrak{a}, |\xi| = 1\} > 0$. For any $\delta > 0$ fixed, there exists an $\epsilon > 0$ such that if $t_\infty - \epsilon < t < t_\infty$, $\frac{\partial u_{\text{ref}}}{\partial \xi_\infty} \geq m/2$ on $B(x_t, \delta)$. This is because $|x_t|$ goes to ∞ and u_{ref} is asymptotic to v. Choose now $\delta = \delta(m/4)$ given by Lemma 6.27, then for t close to t_∞, we obtain

$$\int_{\mathfrak{a}^+} \frac{\partial u_{\text{ref}}}{\partial \xi_\infty} e^{-\nu_t} dx \geq m/4 > 0.$$

□

Combined with the obstruction proved earlier, it gives our necessary and sufficient condition for the existence of a Kähler-Einstein metric.

Assume now $t_\infty < 1$, then we can write

$$\frac{t_\infty}{t_\infty - 1}(\text{bar}_{DH}(2P^+) - 4\rho)(\xi_\infty) = (v - 4\rho)(\xi_\infty),$$

or

$$\left(4\rho + \frac{t_\infty}{1 - t_\infty}(-\text{bar}_{DH}(2P^+) + 4\rho)\right)(\xi_\infty) = v(\xi_\infty).$$

The function $t \mapsto \frac{t}{1-t}$ is strictly increasing and its image is $[0, \infty[$, and $4\rho \in \text{Int}(2P)$. Besides, since v is the support function of $2P$, the value $v(\xi_\infty)$ is attained by $\langle m, \xi_\infty \rangle$ if and only if m is in the supporting hyperplane of $2P$ defined by ξ_∞. We deduce that necessarily t_∞ is the unique value of t for which

$$4\rho + \frac{t}{1 - t}(-\text{bar}_{DH}(2P^+) + 4\rho) \in \partial(2P^+ + \overline{-\Xi}),$$

if it exists. If it does not exist, then $t_\infty = 1$.

Combining this with the upper bound on $R(X)$, we have proved

Theorem 6.31. *The greatest Ricci lower bound of a smooth and Fano group compactification X is*

$$R(X) = \sup\left\{t; \frac{t}{1 - t}(-\text{bar}_{DH}(2P^+) + 4\rho) + 4\rho \in 2P^+ + \overline{-\Xi}\right\}.$$

If this case happens, $R(X) = 1$ with no Kähler-Einstein metrics means $\text{bar}_{DH}(P^+) \in \partial(4\rho + \Xi)$.

6.6 Examples

6.6.1 Rank one examples

For the two rank one examples, which are the wonderful compactifications of $\text{SL}_2(\mathbb{C})$ and of $\text{PGL}_2(\mathbb{C})$, we know that there exists a Kähler-Einstein metric, because they are homogeneous Fano manifolds. We can check that our criterion is satisfied in this situation.

Let us first deal with $\mathbb{P}^3$ which is the wonderful compactification of $\mathrm{PGL}_2(\mathbb{C})$. We can identify $\mathfrak{a}$ with $\mathbb{R}$ and the unique positive root with the multiplication by one. Then the polytope P is $[-2,2]$, P^+ is $[0,2]$, and $2\rho = 1$. The barycenter with respect to the Duistermaat-Heckman measure is then

$$\begin{aligned} \mathrm{bar}_{DH}(P^+) &= \int_0^2 x^3 dx \left(\int_0^2 x^2 dx\right)^{-1} \\ &= 3/2 \\ &> 1 \end{aligned}$$

Consider now the quadric which is the wonderful compactification of $\mathrm{SL}_2(\mathbb{C})$. Then with the same identifications, $P^+ = [0, \frac{3}{2}]$, so

$$\begin{aligned} \mathrm{bar}_{DH}(P^+) &= \int_0^{3/2} x^3 dx \left(\int_0^{3/2} x^2 dx\right)^{-1} \\ &= 9/8 \\ &> 1 \end{aligned}$$

6.6.2 Rank two examples

We computed numerically (with scilab) the barycenter with respect to the Duistermaat-Heckman measure of P^+ for some compactifications of irreducible rank two groups. For all cases we computed the coordinates in the basis given by simple roots, and chose a realization of the root systems in the euclidean plane to determine the barycenters. The barycenter can also be computed exactly, either by hand or with another computer program. We computed the exact values for the wonderful compactification of $\mathrm{PSL}_3(\mathbb{C})$ and for the non Kähler-Einstein example.

Let us give some details about the three rank two root systems, and the results we obtained.

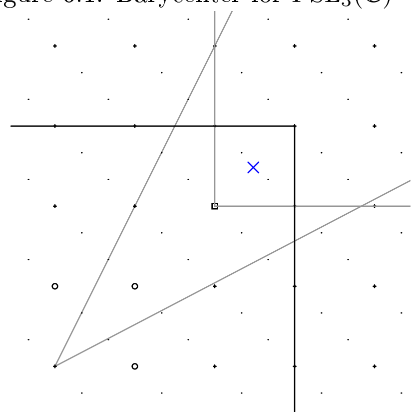
Root system A_2

For the root system $\Phi = A_2$, denote by α_1 and α_2 the simple roots. There are three positive roots: α_1, α_2 and $\alpha_1+\alpha_2$. Choosing to realize A_2 in the unit circle in the euclidean plane, *i.e.* taking $\alpha_1 = (1,0)$ and $\alpha_2 = (-1/2, \sqrt{3}/2)$, we have, for $p = x\alpha_1 + y\alpha_2$,

$$\prod_{\alpha\in\Phi^+} (\alpha(p))^2 = (x - y/2)^2(-x/2 + y)^2(x/2 + y/2)^2.$$

We computed the barycenter by computing first the integrals of 1, x and y on P^+ with respect to the measure with potential $\prod_{\alpha\in\Phi^+}(\alpha(p))^2$ against the

Figure 6.1: Barycenter for $\overline{\mathrm{PSL}_3(\mathbb{C})}^{\mathrm{wond}}$



Lebesgue measure. We used the function int2d in scilab, taking for triangulation the triangles with vertices the origin and other vertices of P^+. We give the coordinates of the barycenters here with a precision estimated by the program to be lower than or of the order of the last digit we write here.

Remark that the coordinates of 2ρ in the basis given by the simple roots are $(2, 2)$ for A_2.

For the wonderful compactification of $\mathrm{PSL}_3(\mathbb{C})$ we obtained

$$(2.4920105\ldots, 2.4920105\ldots)$$

as the coordinates of $\mathrm{bar}_{DH}(P^+)$, so $\mathrm{PSL}_3(\mathbb{C})$ admits a Kähler-Einstein metric. We also computed the exact value in this case, which is

$$\left(\frac{24641}{9888}, \frac{24641}{9888}\right).$$

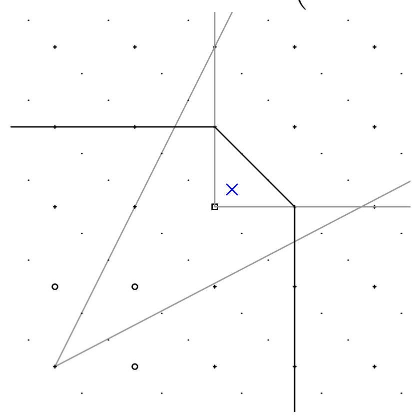
Figure 6.1 gives a representation of $P^+ + (-\overline{\Xi})$ in the plane with coordinates in α_1 and α_2 as abscissas and ordinates. The cone starting from the origin is the positive Weyl chamber (remark that in this representation the Killing form does *not* agree with the euclidean product). The point 2ρ is represented as a little square and the zone in which the barycenter has to be to ensure the existence of a Kähler-Einstein metric is the intersection of the cone starting from 2ρ and P^+. The barycenter is represented by the cross.

Consider now the blow up of the wonderful compactification of $\mathrm{PSL}_3(\mathbb{C})$ at the closed orbit. We obtained the coordinates

$$(2.2169041\ldots, 2.2169041\ldots)$$

for the barycenter (see Figure 6.2), so this manifold again admits a Kähler-Einstein metric.

Figure 6.2: Barycenter for $\mathrm{Bl}\left(\overline{\mathrm{PSL}_3(\mathbb{C})}^{\mathrm{wond}}\right)$



Root system B_2

To realize $\Phi = B_2$ in the euclidean plane, one can choose $\alpha_1 = (1, 0)$ and $\alpha_2 = (-1, 1)$ as simple roots. The other positive roots are $\alpha_1+\alpha_2$ and $2\alpha_1+\alpha_2$. We then compute, for $p = x\alpha_1 + y\alpha_2$,

$$\prod_{\alpha\in\Phi^+} (\alpha(p))^2 = x^2y^2(x-y)^2(-x+2y)^2.$$

We can write here $2\rho = 4\alpha_1 + 3\alpha_2$.

We have described in Chapter 4 three examples of toroidal Fano compactifications of groups of type B_2. The first is the wonderful compactification of $\mathrm{SO}_5(\mathbb{C})$. The coordinates of $\mathrm{bar}_{DH}(P^+)$ in the basis given by the simple roots are (see Figure 6.3)

$$(4.4825505\ldots, 3.3410082\ldots)$$

so this manifold admits a Kähler-Einstein metric.

The second is the wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$. The coordinates of the barycenter are

$$(4.3538528\ldots, 3.0646302\ldots).$$

The corresponding figure is Figure 6.4 and the manifold admits a Kähler-Einstein metric.

Finally, the blow up of the wonderful compactification of $\mathrm{Sp}_4(\mathbb{C})$ does *not* satisfy the criterion. It does not admit any Kähler-Einstein metric. Indeed the coordinates of the barycenter are

$$(4.1525897\ldots, 2.9089329\ldots)$$

so the barycenter is not in the right zone (see Figure 6.5). In this case we also computed the exact values for the coordinates of the barycenter:

$$\left(\frac{278037566905}{66955221696}, \frac{3043253830}{1046175339}\right).$$

Figure 6.3: Barycenter for $\overline{\mathrm{SO}_5(\mathbb{C})}^{\mathrm{wond}}$

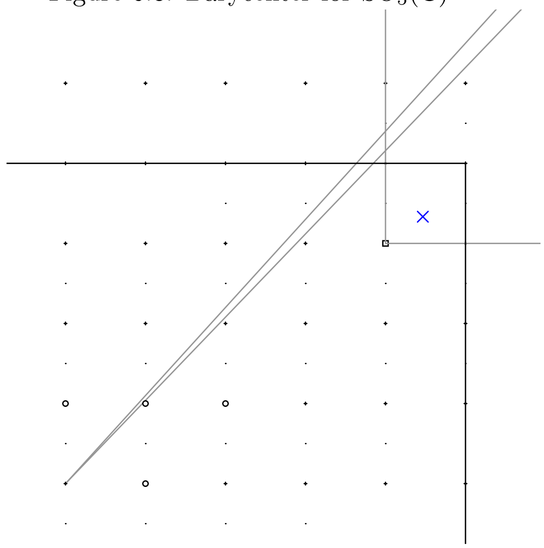


Figure 6.4: Barycenter for $\overline{\mathrm{Sp}_4(\mathbb{C})}^{\mathrm{wond}}$

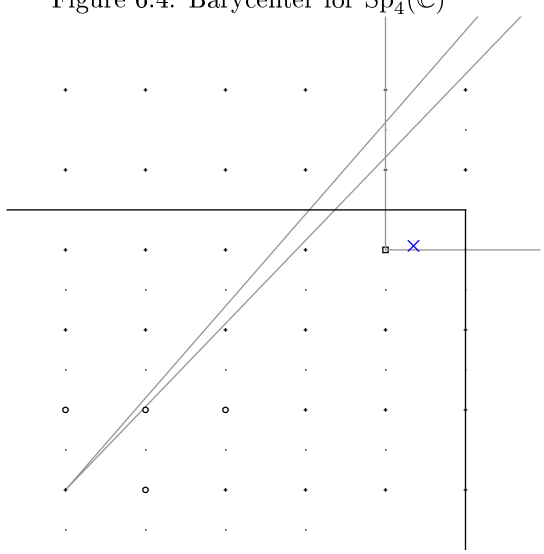
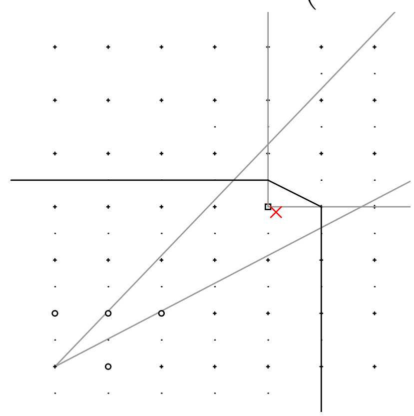


Figure 6.5: Barycenter for $\mathrm{Bl}\left(\overline{\mathrm{Sp}_4(\mathbb{C})}^{\mathrm{wond}}\right)$



This allows to compute exactly the greatest Ricci lower bound for this manifold, which is

$$\frac{1046175339}{1236719713} \simeq 0.8459\ldots$$

Root system G_2

For the root system G_2, we had only one toroidal example, which is the wonderful compactification of the group G_2. In this case we can realize the simple roots in the euclidean plane as $\alpha_1 = (\sqrt{3}/3, 0)$ and $\alpha_2 = (-\sqrt{3}/2, 1/2)$. The other positive roots are $\alpha_1 + \alpha_2$, $2\alpha_1 + \alpha_2$, $3\alpha_1 + \alpha_2$ and $3\alpha_1 + 2\alpha_2$. We can thus compute, for $p = x\alpha_1 + y\alpha_2$,

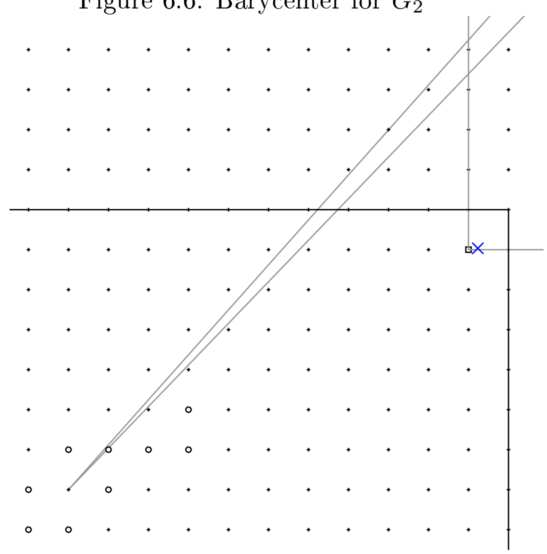
$$\prod_{\alpha \in \Phi^+} (\alpha(p))^2 = \left(\frac{x}{3} - \frac{y}{2}\right)^2 \left(-\frac{x}{2} + y\right)^2 \left(-\frac{x}{6} + \frac{y}{2}\right)^2 \left(\frac{x}{6}\right)^2 \left(\frac{x}{2} - \frac{y}{2}\right)^2 \left(\frac{y}{2}\right)^2.$$

Figure 6.6 shows the barycenter, whose coordinates with respect to the simple roots are

$$(10.260455\ldots, 6.0448053\ldots).$$

This manifold admits a Kähler-Einstein metric. Indeed, the coordinates of 2ρ are $(10, 6)$.

Figure 6.6: Barycenter for $\overline{G_2}^{\mathrm{wond}}$



Appendix A

Alpha invariants of toric line bundles

This is the text from [Del15], published in *Annales Polonici Mathematici*.

Introduction

The α-invariant of a line bundle L on a complex manifold X is an invariant measuring the singularities of the non-negatively curved singular hermitian metrics on L. It was introduced by Tian in the case of the anticanonical bundle on a Fano manifold. Tian showed in [Tia87] that if the α-invariant of the anticanonical bundle is strictly greater than $\frac{n}{n+1}$, then the Fano manifold admits a Kähler-Einstein metric.

The Yau-Tian-Donaldson conjecture asserts in general that X admits an extremal metric in $c_1(L)$ if and only if the line bundle L is K-stable. It was proved in [CDS15a, CDS15b, CDS15c, Tia] that it holds when L is the anticanonical bundle. In particular (as it was shown also in [OS12]), if the α-invariant of the anticanonical bundle is greater than $\frac{n}{n+1}$, then the anticanonical bundle is K-stable. Dervan [Der] gave a similar condition of K-stability for a general line bundle, involving again its α-invariant. This is one motivation to compute explicitly the α-invariants of line bundles when possible.

In [CS08], Chel'tsov and Shramov computed for example the α-invariant of the anticanonical bundle for many Fano manifolds of dimension three. In higher dimensions, Song [Son05] proved a formula giving the α-invariant of the anticanonical bundle on a toric Fano manifold in terms of its polytope. The only toric manifolds satisfying Tian's criterion are the symmetric toric manifolds. Batyrev and Selivanova [BS99] proved first that their α-invariant was one, so that they admit a Kähler-Einstein metric. Wang and Zhu [WZ04] fully settled the question of the existence of Kähler-Einstein metrics on toric Fano manifolds, and an illustration that Tian's criterion is only a sufficient condition can be found in the toric world [NP11].

The α-invariant of a line bundle L is strongly related to the log canonical thresholds (lct) of metrics on L. The log canonical threshold was initially an algebraic invariant defined for ideal sheaves, but it was shown to coincide with the complex singularity exponent and Demailly defines the log canonical threshold of any non-negatively curved singular hermitian metric on a line bundle in [CS08] for example.

One of the main examples of computation of log canonical threshold is in the case of monomial ideals. Howald carried out the computation of the lct of such an ideal in terms of its Newton polygon [How01]. One can find in Guenancia [Gue12] an analytic proof of this result, generalized to compute the lct of an ideal generated by a "toric" psh function on a neighborhood of $0 \in \mathbb{C}^n$, i.e. a function invariant under rotation in each coordinate.

Since the only smooth affine toric manifolds without torus factor are isomorphic to $\mathbb{C}^n$, the computation of Guenancia in fact gives the log canonical threshold of any invariant metric on an affine smooth toric manifold, as we explain in Section A.2.

In this note, we give a formula for the α-invariant of any line bundle L on a compact smooth toric manifold in terms of its polytope. We also compute the log canonical threshold of any invariant non-negatively curved singular metric on L.

Remark A.1. After this article was accepted, the author was informed that other authors computed similar invariants using other methods (H. Li, Y. Shi, Y. Yao [LSY15], and F. Ambro [Amb]).

A.1 Line bundles on smooth toric manifolds

A.1.1 Toric manifolds

Let us recall some basic facts about toric varieties (see [Ful93], [Oda88], [CLS11]).

Let $T = (\mathbb{C}^*)^n$ be an algebraic torus. Denote its group of characters by M, which is isomorphic to $\mathbb{Z}^n$ through the choice of a basis, and let $M_\mathbb{R} := M \otimes \mathbb{R} \simeq \mathbb{R}^n$. The dual N of M consists of the one parameter subgroups of T, and we let also $N_\mathbb{R} := N \otimes \mathbb{R} \simeq \mathbb{R}^n$.

We denote by $T_c \simeq (S^1)^n$ the compact torus in T.

Considering only cones for the toric setting, we will call $\sigma \subset N_\mathbb{R}$ a *cone* if σ is a convex cone generated by a finite set of elements of N. The *dual cone* $\sigma^\vee$ is defined as

$$\sigma^\vee = \{x \in M_\mathbb{R} | \langle x, y\rangle \geq 0 \ \forall y \in \sigma\}.$$

A *fan* Σ consists of a finite collection of cones $\sigma \subset N_\mathbb{R}$ such that every cone is *strongly convex* (*i.e.* $\{0\}$ is a face of σ), the faces of cones in Σ are in Σ and the intersection of two cones in Σ is a union of faces of both. The support of Σ is $|\Sigma| := \bigcup_{\sigma\in\Sigma} \sigma \subset N_\mathbb{R}$.

Recall that a fan Σ in $N_{\mathbb{R}}$ determines a toric variety X_Σ, that is, a normal T-variety with an open and dense orbit isomorphic to T, and every toric variety is obtained this way.

By the orbit-cone correspondence [CLS11, Theorem 3.2.6], a maximal cone σ of Σ corresponds to a fixed point z_σ in X_Σ. Also, a one-dimensional cone ρ in Σ corresponds to a prime invariant divisor D_ρ of X_Σ, and these divisors generate the group of Weil divisors of X_Σ. Let ρ be such a cone, then we denote by u_ρ the primitive vector in N generating this ray. We will denote by $\Sigma(r)$ the set of r-dimensional cones in Σ.

Many properties of X_Σ can be read off from the fan. For example, X_Σ is smooth if and only if every cone in the fan Σ is generated by part of a basis of N. We will call a cone *smooth* if it satisfies this condition. The variety X_Σ is complete if and only if $|\Sigma| = N_{\mathbb{R}}$.

We will assume in general in the following that either $|\Sigma| = N_{\mathbb{R}}$ or that Σ is given by a strongly convex, full dimensional cone σ and its faces, in which case we will denote X_σ the corresponding (affine) toric variety.

A.1.2 Line bundles

Recall that a line bundle L on a G-variety X is called *linearized* if there is an action of G on L such that for any $g \in G$ and $x \in X$, g sends the fiber L_x to the fiber $L_{g\cdot x}$ and the map defined this way between L_x and $L_{g\cdot x}$ is linear.

To a T-linearized line bundle L on X_Σ is associated a set of characters v_σ, for $\sigma \in \Sigma(n)$. We define v_σ as the opposite of the character of the action of T on the fiber over the fixed point z_σ.

This defines the support function g_L of L, which is a function on the support $|\Sigma|$ of Σ, linear on each cone, which takes integral values at points of N, by $x \mapsto \langle v_\sigma, x\rangle$ for $x \in \sigma$.

Another equivalent data is the Weil divisor D_L associated to L, which is related to g_L by the following: $D_L = -\sum_\rho g_L(u_\rho) D_\rho$.

If L is effective, then to L is associated a polytope P_L in $M_{\mathbb{R}}$. This polytope can be defined as

$$P_L = \{m \in M_{\mathbb{R}} | g_L(x) \leq \langle m, x\rangle \ \forall x \in |\Sigma|\}.$$

The properties of the line bundle can be read off from the polytope or the support function. In particular, we can associate to each point of $P_L \cap M$ a global section of L, and the collection of these sections form a basis of the space of algebraic sections of L. Recall also the following, where we assume that $|\Sigma| = N_{\mathbb{R}}$.

Proposition A.2. *[CLS11, Theorem 6.1.7] The following are equivalent:*
- *L is nef*
- *L is generated by global sections*
- *$\{v_\sigma\}$ is the set of vertices of P_L*
- *g_L is concave.*

Proposition A.3. *[CLS11, Lemma 9.3.9] L is big iff P_L has nonempty interior.*

Proposition A.4. *[CLS11, Lemma 6.1.13] The line bundle L is ample iff g_L is concave and $v_\sigma \neq v_{\sigma'}$ whenever $\sigma \neq \sigma' \in \Sigma(n)$.*

Example A.5. The anticanonical divisor $-K_{X_\Sigma}$ on a toric manifold is given by $-K_{X_\Sigma} = \sum_\rho D_\rho$. It is always big on a toric manifold.

A.1.3 Non-negatively curved singular metrics on line bundles

Potential on the torus

Let L be a T-linearized line bundle on X_Σ.

Recall that any linearized line bundle on $T \simeq (\mathbb{C}^*)^n$ is trivial. Fix an invariant trivialization s of L on T.

Given a hermitian metric h on the line bundle L, we denote by φ_h the local potential of h on T, which is the function on T defined by:

$$\varphi_h(z) := -\ln(||s(z)||_h).$$

The local potentials of a smooth hermitian metric are smooth. We will work here with singular metrics, whose local potential are *a priori* only in L^1_{loc}. A singular hermitian metric h is said to have non negative curvature (in the sense of currents) if and only if every local potential of h is a psh function.

A T_c-invariant function φ on T is determined by a function f on $N_\mathbb{R}$, identified with the Lie algebra of T_c, through the equivariant isomorphism:

$$T_c \times N_\mathbb{R} \longrightarrow T;\ ((e^{i\theta_j})_j, (x_j)_j) \mapsto (e^{x_j + i\theta_j})_j.$$

Furthermore, φ is psh if and only if f is convex.

So to a non negatively curved, T_c-invariant metric h on L is associated a convex function f_h, which is the function on $N_\mathbb{R}$ determined by φ_h.

Behavior at infinity of the potentials

Definition A.6. Let L be a nef line bundle on X_Σ. The function $f_L : x \mapsto -g_L(-x)$ is a convex function on $N_\mathbb{R}$, and it is the potential of a continuous, T_c-invariant, non negatively curved metric on L called the Batyrev-Tschinkel metric (see [Mai00]), which we denote by h_L.

Proposition A.7. *The map $h \mapsto f_h$ defines a bijection between the singular hermitian T_c-invariant metrics on L with non-negative curvature, and the convex functions on $N_\mathbb{R}$, such that there exists a constant C with $f_h \leq f_L + C$ on $N_\mathbb{R}$.*

Proof. See also [BB13, Proposition 3.3]. Let h be a singular hermitian T_c-invariant metrics on L with non-negative curvature. Write $h = e^{-v} h_L$, and let ω_L be the curvature current of h_L. Then v is a ω_L-psh function on X. In particular, v is bounded from above on X. Denote by u the convex function on $\mathbb{R}^n$ associated to the T_c-invariant function $v|_T$. Then we see that $f_h(x) - f_L(x) = u(x)$ is bounded above on $N_{\mathbb{R}}$.

Conversely, the standard fact that a psh function, which is bounded from above, extends uniquely over an analytic set, allows one to extend $u := f - f_L$ to an ω_L-psh function on the whole of X if f satisfies the condition of the proposition. $\square$

A.2 Log canonical thresholds

A.2.1 Definition

Let X be a compact complex manifold, and L a line bundle on X. Let h be a singular hermitian metric on L. We recall the definition of the log canonical threshold of h (see the appendix of [CS08]).

Definition A.8. Let $z \in X$. The complex singularity exponent $c_z(h)$ of h at z is the supremum of the real $c > 0$ such that $e^{-2c\varphi}$ is integrable in a neighborhood of z, where φ is a local potential of h near z.

Definition A.9. The log canonical threshold $\mathrm{lct}(h)$ of h is defined as

$$\mathrm{lct}(h) = \inf_{z \in X} c_z(h).$$

A.2.2 Newton body of a function

Definition A.10. Let σ be a cone. Let f be a function defined on $N_{\mathbb{R}}$. Define the Newton body of f on σ as

$$N_\sigma(f) = \{m \in M_{\mathbb{R}}; f(x) - \langle m, x \rangle \geq O(1),\ \forall x \in \sigma\}.$$

If $\sigma = N_{\mathbb{R}}$ we will write $N(f)$.

The following properties of the Newton body will be useful.

Proposition A.11. *For any function f, $N_\sigma(f)$ is convex, and*

$$N_\sigma(f) = N_\sigma(f) - \sigma^\vee.$$

If f is convex, then for any $y \in N_{\mathbb{R}}$,

$$N_\sigma(f) = \{m \in M_{\mathbb{R}}; f(t) - \langle m, t \rangle \geq O(1),\ \forall t \in y + \sigma\}.$$

Proof. The first two properties are trivial. Let us briefly prove the last statement.

Let m be in the right-hand set, *i.e.* $\{f(t) - \langle m, t \rangle \geq O(1)\ \forall t \in y + \sigma\}$. Let $x = t - y \in \sigma$ for $t \in y + \sigma$. By convexity, $f(x+y) \leq \frac{1}{2}(f(2x) + f(2y))$ so we get

$$f(2x) \geq 2f(x+y) - f(2y) = 2f(t) - f(2y)$$

Subtracting $\langle m, 2x\rangle$ gives

$$f(2x) - \langle m, 2x\rangle \geq 2(f(t) - \langle m, t\rangle) + (2\langle m, y\rangle - f(2y)).$$

The right hand side is the sum of a lower-bounded function of $t \in y + \sigma$ and a constant, so the left hand side is a lower-bounded function of $x \in \sigma$.

This shows one inclusion and the other is proved by a similar argument. □

Given a non negatively curved T_c-invariant metric h on L, we define the associated convex subset P_h of $M_{\mathbb{R}}$, as the Newton body of f_h.

Proposition A.12.
- *For the Batyrev-Tschinkel metric h_L, we recover the polytope P_L.*
- *For any T_c-invariant, non-negatively curved metric h on L, $P_h \subset P_L$.*
- *If h is smooth, we also have $P_h = P_L$*

Proof. For the first statement, observe that $m \in P_L$ if and only if for any cone $\sigma \in \Sigma$, for all $x \in \sigma$, $g_L(x) = \langle v_\sigma, x\rangle \leq \langle m, x\rangle$. This inequality is equivalent to $-\langle v_\sigma, x\rangle + \langle m, x\rangle \geq 0$ and since the functions involved are linear, it is satisfied for all $x \in \sigma$ if and only if $-\langle v_\sigma, x\rangle + \langle m, x\rangle$ is bounded below on σ. Since $f_L(-x) = -g_L(x) = -\langle v_\sigma, x\rangle$ for $x \in \sigma$, we get that $m \in P_L$ if and only if for every cone $\sigma \in \Sigma$, the function $f_L(-x) - \langle m, -x\rangle$ is bounded below on σ. Finally, this can be translated as: for every cone $\sigma \in \Sigma$, the function $f_L(y) - \langle m, y\rangle$ is bounded below on $-\sigma$. To conclude, we note that $N(f_L) = \bigcap_\sigma N_{-\sigma}(f_L)$.

The second statement is an easy consequence of the first and Proposition A.7 since whenever two functions f and g satisfy $f \leq g + C$ for a constant C, we have trivially $N_\sigma(f) \subset N_\sigma(g)$.

For the last statement, remark that in this case, $f_h - f_L$ extends to a continuous function on X_Σ, so we have $f_L - C \leq f_h \leq f_L + C$ for some constant C. The same property of Newton bodies allows one to conclude. □

A.2.3 Integrability condition

The first result on log canonical thresholds on toric varieties was the computation by Howald [How01] in the case of monomial ideals. Guenancia gave an analytic proof of this result, extending the computation to the case of non algebraic psh functions. The key ingredient in this analytic version is the following integrability condition.

Proposition A.13. *(see [Gue12]) Let σ be a smooth cone of maximum dimension. Let f be a convex function on $N_{\mathbb{R}}$. Then e^{-f} is integrable on all translates of σ if and only if* $0 \in \mathrm{Int}(N_\sigma(f))$.

This is essentially the result in Guénancia [Gue12] because any smooth affine toric manifold with no torus factor is isomorphic to $\mathbb{C}^n$. However we describe the change of variables used precisely, to use it later in the compact case.

Proof. Choose a basis of N formed by the generators of the extremal rays of σ, then define S_σ to be the isomorphism from N to $\mathbb{Z}^n$ sending the chosen basis to the canonical basis of $\mathbb{Z}^n$.

Let f be a function on $N_\mathbb{R}$, and g the function on $\mathbb{R}^n$ such that $f = g \circ S_\sigma$. Then from the definition of Newton body we have $N_\sigma(f) = S_\sigma^*(N_D(g))$, where S_σ^* is the dual isomorphism from $\mathbb{Z}^n$ to M and D is the cone generated by the canonical basis of $\mathbb{Z}^n$.

Using the change of variables, e^{-f} is integrable on all translates of σ if and only if $e^{-f\circ S_\sigma^{-1}}$ is integrable on all translates of D. Apply [Gue12, Proposition 1.9] to the concave function $-f \circ S_\sigma^{-1}$. This proves that we have integrability if and only if $0 \in \mathrm{Int}(N_D(f \circ S_\sigma^{-1}))$. Using S_σ^*, which is linear, this indeed translates to $0 \in \mathrm{Int}(N_\sigma(f))$.

Remark that the statement in [Gue12, Proposition 1.9] only mentions integrability on D, but the equivalence with integrability on all translates is easily derived from Proposition A.11. □

A.2.4 lct on an affine smooth toric manifold

Proposition A.14. *Let σ be a smooth cone of maximum dimension, X_σ the corresponding smooth affine toric manifold. Let L be a linearized line bundle on X_σ, and h a T_c-invariant metric with non-negative curvature. Then*

$$\mathrm{lct}(h) = \sup\{c > 0 | cv_\sigma \in \mathrm{Int}(N_{-\sigma}(cf_h)) - S_\sigma^*(1, \dots, 1)\}.$$

Proof. The change of variables for cones S_σ in the proof of Proposition A.13 gives (by [CLS11, Theorem 3.3.4]) an equivariant isomorphism between X_σ and $\mathbb{C}^n$, which we denote again by S_σ.

Any linearized line bundle on $\mathbb{C}^n$ is trivial, so L admits a global equivariant trivialization t on X_σ. Remark that, at the fixed point z_σ, we have $g \cdot t(z_\sigma) = -v_\sigma(t(z_\sigma))$ by definition of v_σ. Restricting to T and remembering that s is an invariant trivialization of L on T, we deduce that up to renormalization by a constant, $t(z) = v_\sigma(z)s(z)$ on T.

We can now look at the potential ψ of h with respect to the trivialization t, and remark that, on T, and if φ denotes the potential of h with respect to s on T, we have $\psi(z) = \langle -v_\sigma, \ln|z|\rangle + \varphi(z)$.

Let $y \in N_\mathbb{R}$. Using again the isomorphism $T_c \times N_\mathbb{R} \simeq T$, we consider $T_c \times (y - \sigma)$ as a subset of T, and denote by C_y the closure of this set in X_σ. Each set C_y is a neighborhood of z_σ in X_σ, and they form a basis of neighborhoods. Observe that the collection of the translates of $-\sigma$ cover $N_\mathbb{R}$ and so the corresponding sets cover X_σ. More precisely, for any point z in X_σ, there is a translate of $-\sigma$ which corresponds to a neighborhood of z.

We consider first the complex singularity exponent of h at z_σ. Suppose $c > 0$ is such that $e^{-2c\psi}$ is integrable in a neighborhood of z_σ. Then it is integrable in a neighborhood C_y. We have first that,

$$\int_{C_y} e^{-2c\psi(z)} dz \wedge d\bar{z} = \int_{T_c \times (y-\sigma)} e^{-2c\psi(z)} dz \wedge d\bar{z}.$$

Recall that $\psi(z) = \langle -v_\sigma, \ln|z|\rangle + \varphi(z)$, and that f is the function on $N_\mathbb{R}$ such that $f(x) = \varphi(e^x)$.

Say we have chosen a basis of N or equivalently of M, and we denote by $(x_i)_{i=1\dots n}$ the coordinates of $x \in N_\mathbb{R}$ along this basis. This determines local holomorphic coordinates $z_i = e^{x_i + i\theta_i}$ on $T \simeq N_\mathbb{R} \times T_c$. Using the fact that $\frac{dz_i}{z_i} \wedge \frac{d\overline{z_i}}{\overline{z_i}} = dx_i \wedge d\theta_i$, and T_c-invariance, we obtain that, up to a constant,

$$\int_{C_y} e^{-2c\psi(z)} dz \wedge d\overline{z} = \int_{y-\sigma} e^{-2c(f(x)+\langle -v_\sigma, x\rangle)} e^{2\sum_i x_i} dx.$$

Since $\sum_i x_i$ is equal to $\langle S_\sigma^*(1,\dots,1), x\rangle$, we conclude by using Proposition A.13 that the complex singularity exponent $c_{z_\sigma}(h)$ is the supremum of the $c > 0$ such that $0 \in \mathrm{Int}(N_{-\sigma}(2c(f + \langle -v_\sigma, \cdot\rangle) - 2\langle S_\sigma^*(1,\dots,1), \cdot\rangle))$.

To obtain a simpler condition, remark that for any function g and positive scalar λ, $N_{-\sigma}(\lambda g) = \lambda N_{-\sigma}(g)$, and that if g_1 and g_2 are two convex functions then $N_{-\sigma}(g_1 + g_2)$ is the Minkowski sum of $N_{-\sigma}(g_1)$ and $N_{-\sigma}(g_2)$.

So we get $c_{z_\sigma}(h) = \sup\{c > 0 | cv_\sigma \in \mathrm{Int}(N_{-\sigma}(cf)) - S_\sigma^*(1,\dots,1)\}$.

Furthermore, for any $c < c_{z_\sigma}(h)$, the Proposition A.13 shows that $e^{-2c\psi}$ is integrable on every C_y for $y \in N_\mathbb{R}$. Observe now that for any point $z \in X_\sigma$, there exists a C_y containing z. So for any point $z \in X_\sigma$, $c_z(h) \geq c_{z_\sigma}(h)$. This concludes the proof of the proposition. □

A.2.5 lct on a compact smooth toric manifold

Theorem A.15. *Let X_Σ be a smooth compact toric manifold, L a linearized line bundle on X_Σ and h a T_c-invariant non-negatively curved metric on L. Then*

$$\mathrm{lct}(h) = \sup\{c > 0 | cP_L \subset \mathrm{Int}(cP_h + P_{-K_{X_\Sigma}})\}.$$

Proof. The compact manifold X_Σ is covered by the affine toric manifolds X_σ, for $\sigma \in \Sigma(n)$. By definition of the log canonical threshold,

$$\mathrm{lct}(h) = \min_{\sigma\in\Sigma(n)} \mathrm{lct}(h|_{Z_\sigma}).$$

Another way to say this is that $\mathrm{lct}(h)$ is the sup of $c > 0$ such that $c \leq \mathrm{lct}(h|_{X_\sigma})$ for all $\sigma \in \Sigma(n)$.

Now this condition means, by Proposition A.14, that for all $\sigma \in \Sigma(n)$,

$$cv_\sigma \in \mathrm{Int}(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1), \cdot\rangle).$$

By Proposition A.11, this is equivalent to the condition that for all $\sigma \in \Sigma(n)$,

$$cv_\sigma + \sigma^\vee \subset \mathrm{Int}(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1), \cdot\rangle).$$

This is further equivalent to the condition that for all $\sigma \in \Sigma(n)$,

$$\bigcap_{\sigma\in\Sigma(n)} (cv_\sigma + \sigma^\vee) \subset \mathrm{Int}(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1), \cdot\rangle).$$

Recall from Proposition A.12 that $\bigcap_{\sigma\in\Sigma(n)}(v_\sigma+\sigma^\vee) = N(f_L) = P_L$, so that the condition can be written:

$$N(cf_L) \subset \bigcap_{\sigma\in\Sigma(n)} \mathrm{Int}(N_{-\sigma}(cf_h + \langle -S_\sigma^*(1,\dots,1),\cdot\rangle) = \mathrm{Int}(N(cf_h + f_{-K_{X_\Sigma}})).$$

Indeed, the support function of the anticanonical bundle is, from Example A.5,

$$f_{-K_{X_\Sigma}}(x) = \langle -S_\sigma^*(1,\dots,1), x\rangle.$$

□

A.3 Alpha-invariant

A.3.1 Log canonical threshold and α-invariant

Let X be a compact Kähler manifold, L a big and nef line bundle on X.

Definition A.16. Assume that a compact group K acts on X, and that L is K-linearized. The alpha invariant $\alpha_K(L)$ of L with respect to the group K is defined as the infimum of the log canonical thresholds of all K-invariant, non negatively curved singular hermitian metrics on L.

The linear systems in a multiple of L give singular metrics on L, that we will call algebraic metrics, in the following way. Let $\delta_1,\dots,\delta_r \in H^0(X, mL)$ be linearly independent sections, and denote by Δ the linear system generated by these. Then it defines an algebraic metric $h_{\Delta/m}$ on L by setting, in any trivialization,

$$||\xi||^2_{h_{\Delta/m}} = \frac{|\xi|^2}{(\sum|\delta_j(z)|^2)^{1/m}},$$

for any $\xi \in L_z$. The local potential $\varphi_{\Delta/m}(z) = \frac{1}{2m}\ln\sum|\delta_j(z)|^2$ is psh.

If Δ is one dimensional, generated by δ, we denote by $h_{\delta/m}$ the corresponding metric.

Recall the following result of Demailly, relating the α-invariant with log canonical thresholds of algebraic metrics:

Theorem A.17. *[CS08, Appendix A] Let K be a compact group, let X be a compact complex K-variety and L a big and nef K-linearized line bundle on X. Then*

$$\alpha_K(L) = \inf_{m\in\mathbb{N}^*}\inf_{\Delta\subset H^0(X,mL),\ \Delta^K=\Delta}\mathrm{lct}(h_{\Delta/m}).$$

One can slightly improve this result, and give the following statement, which is only given in the case of a trivial group K by Demailly.

Corollary A.18. *Let K be a compact group, let X be a compact complex K-variety and L a big and nef K-linearized line bundle on X. Then*

$$\alpha_K(L) = \inf_{m\in\mathbb{N}^*}\inf_{\Delta\in\mathrm{Irr}(H^0(X,mL))}\mathrm{lct}(h_{\Delta/m}),$$

where $\mathrm{Irr}(H^0(X,mL))$ *denotes the set of all irreducible K-subrepresentations of* $H^0(X,mL)$.

Proof. Let Δ be a K-invariant subspace of $H^0(X, mL)$, then $\Delta = \Delta_1 \oplus \cdots \oplus \Delta_s$ with Δ_i irreducible subspaces. For all i, one can choose a basis δ_j^i of Δ_i. Together they form a basis of Δ and we can obtain the metric h_Δ this way.

In particular, $\varphi_{\Delta/m}(z) = \frac{1}{2m}\ln \sum_i \sum_j |\delta_j^i(z)|^2$. Since the logarithm is increasing we can write

$$\varphi_{\Delta/m}(z) \geq \frac{1}{2m}\ln \sum |\delta_j^1(z)|^2 = \varphi_{\Delta_1/m}(z).$$

This implies, by elementary properties of the complex singularity exponent, [DK01, 1.4] that $\mathrm{lct}(h_{\Delta/m}) \geq \mathrm{lct}(h_{\Delta_1/m})$.

We conclude that the log canonical threshold of a metric associated to a K-invariant linear system is greater than the log canonical threshold of at least one metric associated to an irreducible linear system, so it is enough to consider only these. □

A.3.2 General formula

Let X_Σ be a smooth compact toric manifold. Let $N(T)$ be the normalizer of T in $\mathrm{Aut}(X_\Sigma)$, and denote by $W = N(T)/T$ the Weyl group obtained from T.

The group $N(T)$ naturally acts on M and since T acts trivially on M, this induces an action of W on M. By duality one also gets an action on N.

From the description of morphisms between toric varieties [CLS11, Theorem 3.3.4], we can see that W is isomorphic to the subgroup of $\mathrm{GL}(N)$ composed of the ρ such that $\rho(\Sigma) = \Sigma$. In particular, W is finite.

Given a subgroup G of W, we denote by T_G the preimage in $N(T)$ of G, and let $K_G := K \cap T_G$. If P is a polytope in $M_\mathbb{R}$ we let P^G be the set of G-invariant points of P.

Finally, if P is a polytope in $M_\mathbb{R}$, we denote by $P(\mathbb{Q})$ the set of rational points in P, i.e. points p such that there exists $m \in \mathbb{N}^*$ with $mp \in M$.

Theorem A.19. *Let L be a T_G-linearized line bundle on X_Σ. Then*

$$\alpha_{K_G}(L) = \inf_{p \in P_L^G(\mathbb{Q})} \sup\{c > 0 | cP_L \subset \mathrm{Int}(cp + P_{-K_{X_\Sigma}})\}.$$

Proof. The Corollary A.18 shows that it is enough to consider algebraic metrics on L associated to K_G-irreducible linear system in a multiple of L.

The T_c-irreducible subrepresentations of $H^0(X_\Sigma, mL)$ are the dimension one subspaces corresponding to integral points of the polytope P_{mL} associated to mL. Recall that $P_{mL} = mP_L$.

Now a K_G-irreducible subrepresentation of $H^0(X_\Sigma, mL)$ is the union of the images by G of a T_c-irreducible representation.

Let p be an integral point in mP_L, and denote by Δ the K_G-irreducible linear system generated by the G-orbit of p.

The potential of $h_{\Delta/m}$ is

$$\varphi_{\Delta/m}(z) = \frac{1}{2m}\ln\left(\sum_{g \in G} |(g \cdot p)(z)|^2\right).$$

By arithmetico-geometric inequality,

$$\varphi_{\Delta/m}(z) \geq \frac{1}{2m}\ln\left|\left(\frac{\sum_{g\in G}(g\cdot p)}{|G|}\right)(z)\right|^2.$$

The right-hand side of this inequality is the potential of the algebraic metric $h_{\frac{\sum_{g\in G}(g\cdot p)}{m|G|}}$ corresponding to the linear system of $H^0(X_\Sigma, m|G|L)$ generated by the section $\sum_{g\in G}(g\cdot p)$.

Using again the fact that the complex singularity exponent is increasing [DK01, 1.4], we get

$$\mathrm{lct}(h_{\Delta/m}) \geq \mathrm{lct}(h_{\frac{\sum_{g\in G}(g\cdot p)}{m|G|}}).$$

We have thus shown that it is enough to compute the log canonical thresholds of algebraic metrics associated to one dimensional G-invariant sublinear systems of multiples of L.

We use Theorem A.15 to conclude. Indeed if $p \in mP_L$ generates a one dimensional G-invariant sublinear system in $H^0(X_\Sigma, mL)$, and $f_{p/m}$ denotes the convex function associated to the potential of the corresponding algebraic metric $h_{p/m}$, we have $N(f_{p/m}) = \{p/m\}$.

Applying Theorem A.15 gives

$$\mathrm{lct}(h_{p/m}) = \sup\{c > 0 | cP_L \subset \mathrm{Int}(cp/m + P_{-K_{X_\Sigma}})\}.$$

Finally, observe that as p and m vary, they describe the set $P_L^G(\mathbb{Q})$ of G-invariant points of P_L with rational coordinates. $\square$

Remark A.20. One can also prove, without the use of Corollary A.18, that we can consider only metrics corresponding to points of P_L (not necessarily with rational coordinates), by considering the expression of the log canonical threshold of any metric.

Indeed, if f is a convex function on $N_\mathbb{R}$, corresponding to a metric h on L, and p is a point in $N(f)$, then the metric h_p associated to the convex function $x \mapsto \langle p, x\rangle$ is also a non-negatively curved metric on L, and $\mathrm{lct}(h_p) \leq \mathrm{lct}(h)$.

A.3.3 Case of the anticanonical line bundle

We assume in this section that $L = -K_{X_\Sigma}$.

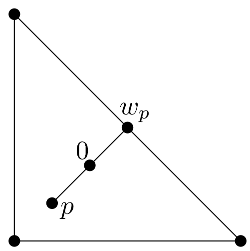
This line bundle admits a natural $\mathrm{Aut}(X)$-linearization, and the polytope associated to this linearization contains the origin in its interior, because $-K_X$ is big.

For any subgroup G of W, let $S_G := \{p \in \partial P_L | g\cdot p = p\ \forall g \in G\}$. If $0 \neq p \in P_L$, let w_p be the point $\partial P_L \cap \{-tp | t \geq 0\}$.

Remark A.21.
- S_G is empty if and only if $\{0\}$ is the only point fixed by G in P.
- If S_W is empty, X_Σ is called symmetric.

Proposition A.22. *Assume that $P_h = \{p\}$ with $0 \neq p \in P_L$. Then*

$$\mathrm{lct}(h) = \frac{|w_p|}{|w_p| + |p|}.$$



Proof. By Theorem A.15 we have

$$\mathrm{lct}(h) = \sup\{c > 0 | cP \subset \mathrm{Int}(cp + P)\}.$$

Consider the half-line starting from p and containing the origin. It intersects ∂P at w_p. Denote by r its intersection with $\partial(p + P)$.

Then it is easy to see that the log canonical threshold of h_p is equal to the quotient of the distance between p and r by the distance between p and w_p. The translation sending 0 to p also sends w_p to r, so $|r - p| = |w_p|$. The result follows. □

Remark A.23. If $P_h = \{0\}$ then $\mathrm{lct}(h) = 1$.

Example A.24. Consider the case $P_h = \{b\}$, where b is the barycenter of the polytope P_L. Then $\mathrm{lct}(h)$ is equal to the greatest lower bound for Ricci curvature $R(X)$, introduced by Székelyhidi [Szé11], and computed for toric manifolds by Li [Li11].

From this formula we recover the previous results of Song and Chel'tsov-Shramov.

Theorem A.25. *[Son05] [CS08, Lemma 6.1] Let X be a smooth Fano toric manifold, and G be a subgroup of W. Then*
- *if S_G is empty, $\alpha_{K_G}(X) = 1$;*
- *else, $\alpha_{K_G}(X) = \frac{1}{1+\max_{p \in S_G} \frac{|p|}{|w_p|}} \leq \frac{1}{2}$.*

Proof. By Theorem A.19, it is enough to consider only the (rational) G-invariant points of P.

The first case follows immediately using Remark A.23.

In the second case, we obtain the formula using Proposition A.22. Indeed, it is enough to consider points p in S_G because if $q \neq 0$ is not in ∂P, and p is the intersection of ∂P with the half line starting from the origin and going through q, then $\mathrm{lct}(h_q) \geq \mathrm{lct}(h_p)$.

Furthermore, $\max_{p \in S_G} \frac{|p|}{|w_p|} \geq 1$ because otherwise if p was such a point at which this maximum was attained and it was < 1 then we would have $\frac{|w_p|}{|p|} > 1$ with $w_p \in S_G$, which is a contradiction. □

A.3.4 Example

We compute the α-invariant of any linearized line bundle on the blow up X of $\mathbb{P}^2$ at one point which we denote X in the following.

Identify N with $\mathbb{Z}^2$. The fan of X has four rays, with generators $u_1 = (1,0)$, $u_2 = (1,1)$, $u_3 = (0,1)$ and $u_4 = (-1,-1)$.

The group W is isomorphic to $\mathbb{Z}/2\mathbb{Z}$ and acts on $M_{\mathbb{R}}$ by exchanging the coordinates $(x,y) \mapsto (y,x)$.

We define the polytope $P(k,l)$ to be the polytope whose vertices are $(0,k)$, $(0,l)$, $(k,0)$ and $(l,0)$, for $k,l \in \mathbb{N}$ with $l > k$. It is easy to see that the polytopes of nef and big divisors are the $P(k,l)$, up to translation by a character. For example, the polytope of the anticanonical bundle is $Q := (-1,-1) + P(1,3)$.

Proposition A.26. *The α-invariant with respect to K_W of the nef and big line bundle corresponding to $P(k,l)$ is equal to* $\inf(\frac{1}{l-k}, \frac{2}{l})$.

Proof. By Theorem A.19, it is enough to consider points (with rational coordinates) in the intersection of $P(k,l)$ with the first diagonal. However, one easily remarks that it is enough to consider only the point $(l/2, l/2)$, similarly to the proof of Theorem A.25.

We want to compute

$$\sup\{c > 0 | cP(k,l) \subset \mathrm{Int}(c(l/2, l/2) + Q)\}.$$

This is of course equal to

$$\sup\{c > 0 | P(k,l) \subset \mathrm{Int}((l/2, l/2) + \frac{1}{c}Q)\}.$$

Observe that $l/2$ is the least positive constant b such that

$$\{(0,l), (l,0)\} \subset (l/2, l/2) + bQ.$$

If $k \geq l/2$, then we have also $\{(0,k), (k,0)\} \subset (l/2, l/2) + l/2Q$, so

$$P(k,l) \subset (l/2, l/2) + l/2Q.$$

Thus $\alpha_{K_W}(P(k,l)) = 2/l$ when $k \geq l/2$.

For the other case, observe that $l - k$ is the least positive constant b such that $(k/2, k/2) \in (l/2, l/2) + bQ$. If $k \leq l/2$, then we have also

$$P(k,l) \subset (l/2, l/2) + (l-k)Q.$$

Thus $\alpha_{K_W}(P(k,l)) = \frac{1}{l-k}$ when $k \geq l/2$. □

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